Math Contest CD Exam Solution
November 12, 2022

Directions: If units are involved, include them in your answer.

1. Find $a+b+c$ if a, b, and c are positive integers satisfying

$$abc + 2ab + 2bc + 2ca + 4a + 4b + 4c = 447$$

Solution. Adding 8 to the above we have

$$\begin{aligned} abc + 2(ab+bc+ca) + 4(a+b+c) + 8 &= 447 + 8 \\ (a+2)(b+2)(c+2) &= 455 = 5 \cdot 7 \cdot 13 \end{aligned}$$

WOLG,

$$a+2 = 5, \quad b+2 = 7, \quad c+2 = 13,$$

or $a+b+c = 19$.

Answer. 19

2. Find the difference between the maximum and the minimum of y satisfying

$$\log_2 x + \frac{12}{\log_2 x} - \log_x y = 6$$

if $2 \le x \le 16$.

Solution. Let M and m be the maximum and the minimum of y satisfying the given equation. Let $X = \log_2 x$ and $Y = \log_2 y$. The given equation becomes

$$X + \frac{12}{X} - \frac{Y}{X} = 6 \quad \Rightarrow \quad Y = X^2 - 6X + 12$$

Since $1 \le X \le 4$, Y attains the maximum $\log_2 M = 1^2 - 6 + 12 = 7$ and the minimum $\log_2 m = 3^2 - 18 + 12 = 3$. Thus the difference is $M - m = 2^7 - 2^3 = 128 - 8 = 120$.

Answer. 120

3. A parallelogram has sides of length 2 and 3. One of its diagonals has length 4. Find the length of the other diagonal.

Solution. Let B and D be endpoints of the diagonal of length 4. Let A and C be the other two vertices of the parallelogram denoted so that $|AB| = |CD| = 2$ and $|AD| = |BC| = 3$. Applying the Law of Cosines to the triangle ABD, we obtain $|BD|^2 = |AB|^2 + |AD|^2 - 2|AB| \cdot |AD| \cos \angle BAD$. Then

$$\cos \angle BAD = \frac{|AB|^2 + |AD|^2 - |BD|^2}{2|AB| \cdot |AD|} = \frac{2^2 + 3^2 - 4^2}{2 \cdot 2 \cdot 3} = -\frac{1}{4}.$$

The angles BAD and ABC are adjacent angles of a parallelogram. Therefore $\angle BAD + \angle ABC = \pi$, which implies that $\cos\angle ABC = -\cos\angle BAD = 1/4$. Applying the Law of Cosines to the triangle ABC, we obtain

$$|AC|^2 = |AB|^2 + |BC|^2 - 2|AB| \cdot |BC| \cos\angle ABC = 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \frac{1}{4} = 10.$$

Thus the diagonal AC has length $\sqrt{10}$.

Answer: $\sqrt{10}$.

4. If $8^x - 8^{-x} = 4$ for a real number x, what is the value of $2^x - 2^{-x}$?

Solution. From the factoring and rewriting

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)\big((a-b)^2 + 3ab\big)$$

we can write

$$8^x - 8^{-x} = (2^x - 2^{-x})\big((2^x - 2^{-x})^2 + 3\big) = 4.$$

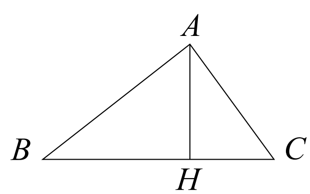
Substituting $X = 2^x - 2^{-x}$, the above become a cubic equation

$$X(X^2+3) = 4 \quad \Leftrightarrow \quad X^3 + 3X - 4 = 0 \quad \Leftrightarrow \quad (X-1)(X^2+X+4) = 0$$

Since $X = 2^x - 2^{-x}$ is a real number, $2^x - 2^{-x} = 1$.

Answer. 1

5. Find the area of $\triangle ABC$ if the perimeter is 30, $AH = 6$, $\overline{AH} \perp \overline{BC}$, and $\overline{AC} \perp \overline{AB}$.



Solution. Let $AB = x$ and $AC = y$. By the Pythagorean theorem, we have

$$x^2 + y^2 = (30 - x - y)^2 \quad \text{or} \quad 0 = 30^2 + 2xy - 60(x+y)$$

By the area of $\triangle ABC$, we also have

$$6(30 - x - y) = xy \quad \text{or} \quad -6(x+y) = xy - 180.$$

So, we get the system of two linear equations for xy and $x + y$:

$$60(x+y) - 2xy = 900 \qquad (1)$$
$$6(x+y) + xy = 180 \qquad (2)$$

Solving the equations, we get $xy = 75$.

The area of $\triangle ABC$ is

$$\frac{1}{2} AB \cdot AC = \frac{75}{2}.$$

Answer. $\frac{75}{2}$

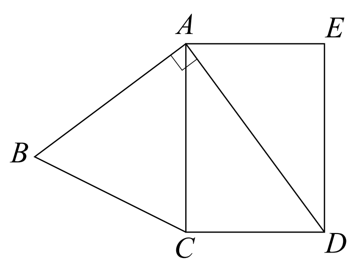
6. Suppose T is a triangle in the plane with sides of length 2, 3 and 4. Let F be the figure that consists of all points of T as well as all points at distance at most 1 from the triangle. Find the perimeter of the figure F.

Solution. The figure F can be cut into 7 pieces: the triangle T, three rectangles with one side of length 1, and three circular sectors of circles of radius 1. Each rectangle shares a side with the triangle T and each sector is centered at a vertex of T. The boundary of F consists of three line segments, each of which is a translation of one side of T, and three circular arcs, which are curvilinear parts of boundaries of the sectors. It follows that the perimeter of F equals $p + s$, where p is the perimeter of the triangle T (equal to 9) and s is the sum of central angles of the sectors.

At each vertex of the triangle T, four pieces of the figure F meet: the triangle itself, two rectangles and one sector. Since the full angle at the vertex equals 2π and every angle of rectangles equals $\pi/2$, it follows that the central angle of the sector equals $\pi - \alpha$, where α is the angle of T at the same vertex. Let α, β and γ be angles of the triangle T. Then the central angles of the sectors add up to $(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 3\pi - (\alpha + \beta + \gamma)$, which is equal to 2π as the sum of angles of any triangle equals π.

Answer: $2\pi + 9$.

7. Consider a rectangle $ACDE$ with $AE = 1$, $AC = \sqrt{3}$. Let B be the point such that $AB = AC$ and $\overline{\mathrm{AB}} \perp \overline{\mathrm{AD}}$. Find the distance between C and $\overline{\mathrm{BD}}$.



Solution. The right triangle $\triangle ABD$ has sides $AB = \sqrt{3}$, $AD = \sqrt{1^2 + (\sqrt{3})^2} = 2$, and

$$BD = \sqrt{(\sqrt{3})^2 + (2)^2} = \sqrt{7}.$$

On $\triangle CAD$, we have

$$\tan(\angle CAD) = \frac{CD}{AC} = \frac{1}{\sqrt{3}},$$

which implies $\angle CAD = 30^\circ$ and $\angle CAB = 60^\circ$. Since $BC = AB = AC = \sqrt{3}$ and $\angle BCD = 150^\circ$ the area of $\triangle BCD$ is

$$\frac{1}{2} BC \cdot CD \sin 150^\circ = \frac{1}{2}\sqrt{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{4},$$

which equals to

$$\frac{1}{2}BD \cdot h = \frac{1}{2}\sqrt{7} \cdot h$$

where h is the distance between C and $\overline{\mathrm{BD}}$. The distance is $\dfrac{\sqrt{3}}{2\sqrt{7}} = \dfrac{\sqrt{21}}{14}$

Answer. $\dfrac{\sqrt{3}}{2\sqrt{7}}$ or $\dfrac{\sqrt{21}}{14}$

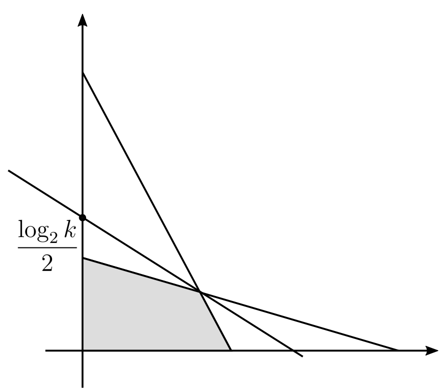
8. Find the maximum of $2^x \cdot 4^y$ provided $\begin{cases} x+3y \leq 5 \\ 2x+y \leq 5 \\ 0 \leq x,\ 0 \leq y \end{cases}$

Solution. The given constraints determine a convex polygon D. Let $k = 2^{x+2y}$. Then $\log_2 k = x + 2y$. We apply Linear Programming to find the maximum y-intercept of the line

$$y = -\frac{1}{2}x + \frac{\log_2 k}{2}$$

when it passes D. As illustrated below the largest possible y-intercept occurs when the line passes the intersection $(2,1)$ of $x+3y=5$ and $2x+y=5$. The maximum of $k = 2^{x+2y}$ is

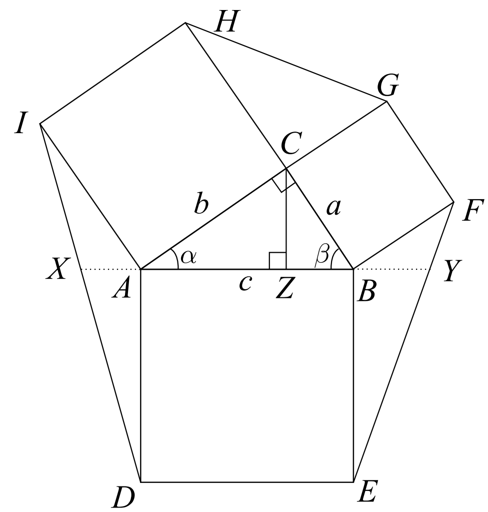
$$k = 2^{2+2} = 16.$$



Answer. 16

9. The hypotenuse of the right triangle has length c and legs have length a and b. On each side of the triangle a square is drawn outside of the triangle. Express the area of the hexagon in terms of a, b, and c, whose vertices are the vertices of these squares which are not vertices of the original triangle?

Solution. Consider the triangle $\triangle ABC$ with $\angle C = 90°$, $\angle A = \alpha$ and $\angle B = \beta$ as in the figure below.



We want to find the areas of triangles $\triangle ADI$ and $\triangle EBF$. Note that $\angle IAD = 180° - \alpha$, $AI = b$, $AD = c$. Besdies, $\sin(180° - \alpha) = \sin\alpha = \frac{a}{c}$. Therefore the area of the triangle is

$$\triangle ADI = \frac{1}{2}bc\sin(180° - \alpha) = \frac{1}{2}bc\sin(\alpha) = \frac{1}{2}b\cancel{c}\frac{a}{\cancel{c}} = \frac{1}{2}ab.$$

Similarly, area of $\triangle EBF$ is equal to $\frac{1}{2}ab$ as well. Moreover, the two congrudent triangles $\triangle BCA$ and $\triangle GCH$ have area $\dfrac{1}{2}ab$.

Consequently, the hexagon consists of three squares and four triangles, each of which has area $\frac{1}{2}ab$. Thus the area is

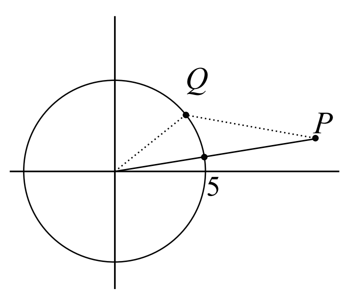
$$a^2 + b^2 + c^2 + 4\frac{1}{2}ab = (a+b)^2 + c^2$$

Answer. $(a+b)^2 + c^2$ or $a^2 + b^2 + 2ab + c^2$

10. For $0 < x < 5$ and $0 < t$, find the minimum value of the following.

$$(x-t)^2 + \left(\sqrt{25 - x^2} - \frac{72}{t}\right)^2$$

Solution. Let D be the given expression. Then $\sqrt{D}$ is the distance PQ between a point $P = \left(t, \dfrac{72}{t}\right)$ on the first quadrant and a point $Q = (x, \sqrt{25 - x^2})$ on a circle of radius 5 centered at the origin.



By the triangle inequality, we have

$$OQ + QP \geq OP = \sqrt{t^2 + \left(\frac{72}{t}\right)^2} \quad \Rightarrow \quad PQ \geq \sqrt{t^2 + \left(\frac{72}{t}\right)^2} - 5$$

The equality (in the last inequality) holds when Q is on the line joining the origin and P. On the other hand, by Arithmetic Mean-Geometric Mean inequality

$$D = PQ^2 \geq \left(\sqrt{t^2 + (\frac{72}{t})^2} - 5\right)^2 \geq \left(\sqrt{2\sqrt{t^2 \cdot (\frac{72}{t})^2}} - 5\right)^2 = \left(\sqrt{2 \cdot 72} - 5\right)^2 = 7^2 = 49,$$

and the equality holds when $t = \sqrt{72}$ (recall that it is assumed that $t > 0$), which again corresponds to the case when Q is on the line joining the origin and P. So the expression D attains the minimum when Q is on the line joining the origin and P and this minimum is 49.

Answer. 49

11. Let a, b, and c be three numbers (not necessarily different) chosen randomly and independently from the set $\{1, 2, 3, 4, 5\}$. Find the probability that the number $ab + c$ is even.

Solution. The sum is even in the following two cases:

I. both ab and c are even. This occurs in the following three subcases: (a- even, b-odd, c-even); (a- odd, b-even, c-even); (a-even, b-even, c-even); which gives

$$2 \cdot 3 \cdot 2 + 3 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 = 12 + 12 + 8 = 32$$

choices.

II. both ab and c are odd. This occurs only if each of the three numbers are odd which gives $3 \cdot 3 \cdot 3 = 27$ choices.

The total number of triples chosen from the set $\{1, 2, 3, 4, 5\}$ is $5 \cdot 5 \cdot 5 = 125$, so the probability is

$$P = \frac{32 + 27}{125} = \frac{59}{125}$$

Answer. $\frac{59}{125}$

12. Let f be a monic polynomial of degree 4 with integer coefficients, and let $g(x) = (x - n)f(x)$ for an integer n. Find n if

   I $g(4) = 13$, $g(9) = 8$

   II $f(-x) = f(x)$

Solution. Condition II implies that $f(x)$ contains only terms with even degree. Let

$$f(x) = x^4 + Ax^2 + B.$$

From condition I we have

$$g(4) = (4-n)(4^4 + 4^2A + B) = 13, \quad g(9) = (9-n)(9^4 + 9^2A + B) = 8 \tag{3}$$

Observe that

$$(4-n)|13 \quad \text{and} \quad (9-n)|8$$

The only integers satisfying the above condition are $n = 5$ and $n = 17$. If $n = 5$, by (3), we have

$$-(256 + 16A + B) = 13 \quad \text{and} \quad (6561 + 81A + B) = 2$$

Eliminating B, we have to have $65A + 6290 = 0$. However, A must be an integer. Indeed $n = 17$ is the desired integer;

$$-(256 + 16A + B) = 1 \quad \text{and} \quad -(6561 + 81A + B) = 1,$$

which yield $A = -97$ and $B = 1295$. We have found $n = 17$.

Answer. $n = 17$

13. How many 9's are there in the decimal expansion of 99999899999^2 ?

Solution. Let $x = 99999899999$. Observe that if we add 100001 to x, we get

$$\begin{array}{r} 99999899999 \\ +100001 \\ \hline 100000000000 \end{array}$$

In other words, $x + 10^5 + 1 = 10^{11}$. From this, we can conclude $x = 10^{11} - 10^5 - 1$. So

$$x^2 = (10^{11} - 10^5 - 1)^2 = 10^{22} + 10^{10} + 1 + 2 \cdot 10^5 - 2 \cdot 10^{16} - 2 \cdot 10^{11}$$

Now,

$$\begin{aligned} &(10^{22} + 10^{10} + 1 + 2 \cdot 10^5) - (2 \cdot 10^{16} + 2 \cdot 10^{11}) \\ &= 10000000000010000200001 - (20000000000000000 + 200000000000) \\ &= 9999979999810000200001 \end{aligned}$$

So the digit 9 appears nine times in the decimal expansion of 99999899999^2.

Answer. 9

14. Find $f(x)$ if $f(2022x + f(0)) = 2022x^2$ for all real numbers x and $f(0) \neq 0$.

Solution. With substitution $t = 2022x$, we have

$$f(t + f(0)) = \frac{t^2}{2022}$$

for all real numbers t.

We translate the function by $f(0)$ (or replacing t by $t - f(0)$) to have

$$f(t) = f\big((t - f(0)) + f(0)\big) = \frac{(t - f(0))^2}{2022} \qquad (4)$$

This quadratic function f is completely determined by $f(0)$. By (4), $f(0)$ is given by

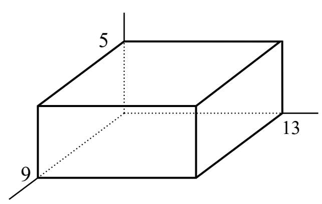
$$f(0) = \frac{(f(0))^2}{2022}.$$

The above impies $f(0) = 2022$ since $f(0) \neq 0$.

Answer. $f(x) = \dfrac{(x - 2022)^2}{2022}$

15. Suppose a rectangular prism is built out of $9 \times 13 \times 5$ unit cubes. Find the number of unit cubes that the main diagonal passes through.

Solution.



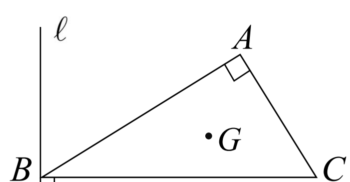
Solution The diagonal passes from one unit cube to another when it intersects one of the planes $x = n$, for $0 < n < 9$, or $y = m$, for $0 < y < 13$ or $z = k$, for $0 < k < 5$. Since the numbers 9, 13 and 5 are mutually coprime, the diagonal intersects those planes one at a time. Hence there are

$$(9 - 1) + (13 - 1) + (5 - 1) = 24$$

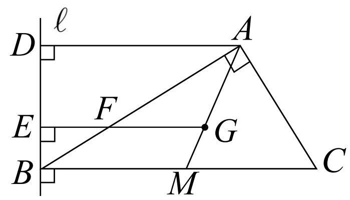
transitions from one unit cube to another. Therefore the number of intersected cubes is 24+1=25.

Answer. 25

16. Consider the triangle $\triangle ABC$ with $AC = 6$, $AB = 8$, and $\overline{\mathrm{AC}} \perp \overline{\mathrm{AB}}$. Let ℓ be the line passing B that is perpendicular to $\overline{\mathrm{BC}}$. Find the distance between ℓ and the centroid G of $\triangle ABC$. (The centroid of $\triangle ABC$ is the point in which the three medians of the triangle intersect)



Solution. Let D and E be perpendicular foots on ℓ from A and G respectively. Let M be the mid point of $\overline{\mathrm{BC}}$, and let F be the intersection of $\overline{\mathrm{EG}}$ and $\overline{\mathrm{AB}}$.



From a pair of similar triangles $\triangle AFG$ and $\triangle ABM$, we also have

$$FG : BM = AG : AM = 2 : 3 \quad \Rightarrow \quad FG = \frac{10}{3}$$

To find EF we use the similarity $\triangle ABC \sim \triangle DAB$. The similarity follows from $\angle ABC = \angle DAB$ (opposite angle) and $\angle BAC = 90° = \angle ADB$. First, we have

$$8 : DA = 10 : 8 \quad \Rightarrow \quad DA = \frac{32}{5}$$

On the other hand, a pair of similar triangles $\triangle BAD \sim \triangle BFE$ implies

$$EF : DA = BF : BA = 1 : 3 \quad \Rightarrow \quad EF = \frac{1}{3}DA = \frac{1}{3}\frac{32}{5} = \frac{32}{15}.$$

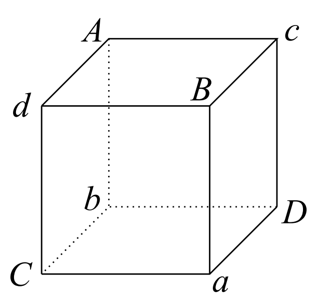
Now, $EG = EF + FG = \dfrac{32}{15} + \dfrac{10}{3} = \dfrac{82}{15}$

Answer. $\dfrac{82}{15}$

17. Let X be the set of 8 vertices of a unit cube. Find the number of one-to-one functions $f : X \to X$ such that the distance between $f(v)$ and v is 1 for all vertices of X.

Solution. Let $X = \{A, B, C, D, a, b, c, d\}$ and label the vertices of a cube as in the figure. If a one-to-one function f satisfies the distance condition, then

$$f \text{ maps upper cases to lower cases and lower cases to upper cases.} \tag{5}$$



First, we can consider the restriction of f on $\{A, B, C, D\}$. Observe that for a given $f(A)$, there are three possible choices for $f(B), f(C)$, and $f(D)$. For example, if $f(A) = b$, then we have

$f(A)$	$f(B)$	$f(C)$	$f(D)$
b	a	d	c
b	c	d	a
b	d	a	c

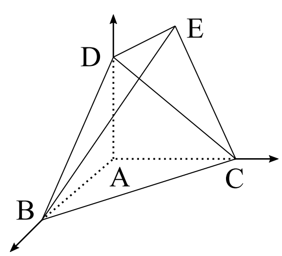
For each of $f(A) = c$ and $f(A) = d$, we also have three cases, and hence there 3×3 functions satisfying the first condition of (5). On the other hand, the restriction f on $\{a, b, c, d\}$ is independent from f on $\{A, B, C, D\}$, and also allows 9 functions. There are $9 \times 9 = 81$ functions satisfying the given condition.

Answer. 81

18. Let A, B, C, D, and E be the points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 2)$ respectively. Find the volume of the polyhedron with edges $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, $\overline{BE}$, $\overline{CD}$, $\overline{CE}$, and $\overline{DE}$.

Solution. The solid consists of two tetrahedrons ABCD and BCDE that share △BCD. The volume of tetrahedron ABCD is

$$1 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$



For the volume of the tetrahedron BCDE, we need to find area of the base △BCD and height. Since △BCD is a equilateral triangle, the area is

$$\frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \sqrt{2} \cdot \sqrt{2} = \frac{\sqrt{3}}{2}.$$

Observe that △BCD is perpendicular to $\overline{\text{DE}}$. The equation of the plane containing △BCD is $x + y + z = 1$ and $\overline{\text{DE}}$ is parallel to $(1, 1, 1)$. Thus the volume of tetrahedron BCDE is

$$\frac{1}{3} \cdot \frac{\sqrt{3}}{2} \cdot DE = \frac{\sqrt{3}}{6} \cdot \sqrt{3} = \frac{1}{2}$$

Now the volume of the entire solid is

$$\frac{1}{6} + \frac{1}{2} = \frac{2}{3}.$$

Answer. $\frac{2}{3}$

19. Let α, β, and γ be the three roots of $x^3 - x - 2 = 0$. Find $\Big((\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\Big)^2$.

Solution. From the condition,

$$\begin{aligned} &\alpha+\beta+\gamma = 0 \\ &\alpha\beta+\beta\gamma+\gamma\alpha = -1 \\ &\alpha\beta\gamma = 2 \end{aligned}$$

we have

$$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = (\gamma)^2 - 4\alpha\beta = \frac{\gamma^3 - 4\alpha\beta\gamma}{\gamma} = \frac{\gamma-6}{\gamma}$$

since $\gamma^3 = \gamma + 2$. Similarly, we also have

$$(\beta-\gamma)^2 = \frac{\alpha-6}{\alpha}, \quad (\gamma-\alpha)^2 = \frac{\beta-6}{\beta}$$

The given expression becomes

$$\Big((\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\Big)^2 = \frac{(\alpha-6)(\beta-6)(\gamma-6)}{\alpha\beta\gamma}$$

On the other hand, plugging $x = 6$ into the equation we have

$$x^3 - x - 2 = (x-\alpha)(x-\beta)(x-\gamma) \quad \Rightarrow \quad 208 = (6-\alpha)(6-\beta)(6-\gamma).$$

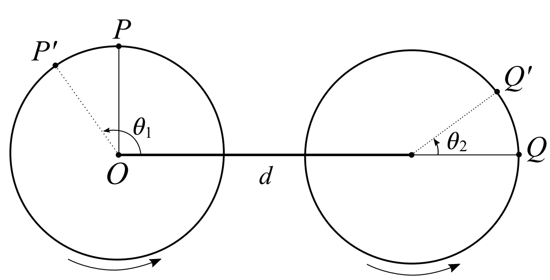
Thus we have

$$\Big((\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\Big)^2 = \frac{-208}{2} = -104.$$

Answer. -104

20. Suppose a bike has wheels with radius 1 ft and the axle distance 3 ft. Consider two rim points on the front and rear wheels of a bike respectively with angle difference 90° (mod 360°). Find the largest distance between these two points while a bike moves along straight line.

Solution. Let P and Q be the points on wheels before moving. Consider the case when P and Q are as in the following figure. To compute the distance, let O be the origin on the plane and let P' and Q' be the points on the wheel at the moment.



Since the angle difference is 90°, we have $\theta_1 - \theta_2 = 90° \pmod{360°}$. The distance between $P'(\cos\theta_1, \sin\theta_1)$ and $Q'(d + \cos\theta_2, \sin\theta_2)$ becomes

$$\begin{aligned}
P'Q' &= \sqrt{(d + \cos\theta_2 - \cos\theta_1)^2 + (\sin\theta_2 - \sin\theta_1)^2} \\
&= \sqrt{(d + \cos\theta_2 - \cos(\theta_2 + 90°))^2 + (\sin\theta_2 - \sin(\theta_2 + 90°))^2} \\
&= \sqrt{(d + \cos\theta_2 + \sin\theta_2)^2 + (\sin\theta_2 - \cos\theta_2)^2} \\
&= \sqrt{\big(d^2 + 1 + 2d(\sin\theta_2 + \cos\theta_2) + 2\cos\theta_2 \sin\theta_2\big) + (1 - 2\cos\theta_2 \sin\theta_2)} \\
&= \sqrt{d^2 + 2 + 2d(\sin\theta_2 + \cos\theta_2)},
\end{aligned}$$

where d be the distance between the centers of two wheels. The sum $\sin\theta_2 + \cos\theta_2$ attains the maximum $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$ at $\theta_2 = 45°$. Thus the largest distance becomes

$$L = \sqrt{3^2 + 2 + 2 \cdot 3(\sqrt{2})} = \sqrt{11 + 2 \cdot \sqrt{18}} = \sqrt{(\sqrt{9} + \sqrt{2})^2} = 3 + \sqrt{2}$$

Analogous observation confirms the same maximum occurs when $\theta_2 - \theta_1 = 90° \pmod{360°}$.

Answer. $3 + \sqrt{2}$ ft