Least-Squares Methods for Computational Electromagnetics

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ABSTRACT

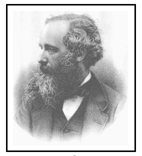
A new approach to a variety of problems related to the system of Maxwell's equations is presented. It is a negative-norm least-squares algorithm based on the ideas from [1]. The numerical experiments included confirm that the new method is a valuable alternative in problems ranging from magnetostatics to the computation of Maxwell eigenvalues.

[1] J.H. Bramble and J.E. Pasciak. *A new approximation technique for div-curl systems.* Math. Comp. (to appear).

1 Electromagnetic Theory

Let Ω be a domain containing linear, isotropic, inhomogeneous material with magnetic permeability μ and electric permittivity ε. Electromagnetic phenomena in Ω can be described by the time harmonic Maxwell equations:

$$\begin{cases} \nabla\times \boldsymbol{h} = \lambda\varepsilon\boldsymbol{e} + \boldsymbol{j} & \text{in } \Omega,\\ \nabla\times \boldsymbol{e} = -\lambda\mu\boldsymbol{h} & \text{in } \Omega,\\ \boldsymbol{e}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega,\\ \mu\boldsymbol{h}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$



where $\boldsymbol{h}, \boldsymbol{e} : \Omega \to \mathbb{C}^3$ are the magnetic and electric fields, $\lambda = i\omega$ is the frequency of propagation and $\boldsymbol{j}$ is the current density. The boundary conditions correspond to Ω surrounded by a perfect conductor. Problems related to these equations arise in various practical applications, but often, due to the large null-space of the *curl* operator, are difficult to solve. The full system (1) can be used to compute the electromagnetic field generated by a given current. When $\boldsymbol{j} = \boldsymbol{0}$ we have the eigenvalue problem, which describes the frequencies that will propagate through the medium. This is important in the design of various structures as waveguides and accelerators. The case of magnetostatics involves only $\boldsymbol{h}$ and is modeling the magnetic fields produced by steady currents. The case of electrostatics involves only $\boldsymbol{e}$ and describes the electric fields produced by stationary source charges.

2 Reformulation of the Eigenvalue Problem

Our method is based on a very weak formulation of the magnetostatic and electrostatic problems. Here we illustrate how it is applied to the eigenvalue problem. Consider the two div-curl systems

$$\begin{cases} \nabla\times\boldsymbol{h} = \varepsilon\boldsymbol{g}_1 & \text{in } \Omega,\\ \nabla\cdot(\mu\boldsymbol{h}) = 0 & \text{in } \Omega,\\ \mu\boldsymbol{h}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad\text{and}\quad \begin{cases} \nabla\times\boldsymbol{e} = \mu\boldsymbol{g}_2 & \text{in } \Omega,\\ \nabla\cdot(\varepsilon\boldsymbol{e}) = 0 & \text{in } \Omega,\\ \boldsymbol{e}\times\boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

These problems are solvable if the source terms $\boldsymbol{g}_k \in \boldsymbol{L}^2(\Omega)$, $k = 1,2$ satisfy certain compatibility conditions. Define the test spaces

$$\boldsymbol{V}_1 = \boldsymbol{H}_0^1(\Omega),\ H_1 = H^1(\Omega),\ \boldsymbol{V}_2 = \boldsymbol{H}^1(\Omega),\ H_2 = H_0^1(\Omega),\ \boldsymbol{Y}_k = \boldsymbol{V}_k \times H_k, k = 1,2\,.$$

The weak formulations of (2) are obtained using integration by parts:

$$\begin{array}{lll} \text{Find } \boldsymbol{h}\in\boldsymbol{L}^2(\Omega) \text{ satisfying} & (\boldsymbol{h},\nabla\times\boldsymbol{v}) + (\mu\boldsymbol{h},\nabla h) = (\varepsilon(\boldsymbol{I}-\boldsymbol{Q}_1)\boldsymbol{g}_1,\boldsymbol{v}) & \text{for all } (\boldsymbol{v},h)\in\boldsymbol{Y}_1\,.\\ \text{Find } \boldsymbol{e}\in\boldsymbol{L}^2(\Omega) \text{ satisfying} & (\boldsymbol{e},\nabla\times\boldsymbol{v}) + (\varepsilon\boldsymbol{e},\nabla h) = (\mu(\boldsymbol{I}-\boldsymbol{Q}_2)\boldsymbol{g}_2,\boldsymbol{v}) & \text{for all } (\boldsymbol{v},h)\in\boldsymbol{Y}_2\,. \end{array} \tag{3}$$

$\boldsymbol{Q}_k$ are $\boldsymbol{L}^2(\Omega)$ projectors related to compatibility, i.e. $\boldsymbol{g}_k$ is compatible data if and only if $\boldsymbol{Q}_k\boldsymbol{g}_k = 0$. The solution operators $\boldsymbol{S}_k : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{L}^2(\Omega)$ are defined by $\boldsymbol{S}_k\boldsymbol{g}_k \equiv \boldsymbol{x}_k$. All eigenfunctions of the original system with nonzero λ satisfy $\boldsymbol{S}_1(\lambda\boldsymbol{e}) = \boldsymbol{h}$ and $\boldsymbol{S}_2(-\lambda\boldsymbol{h}) = \boldsymbol{e}$, i.e.

$$\boldsymbol{B}\begin{pmatrix}\boldsymbol{e}\\ \boldsymbol{h}\end{pmatrix} \equiv \begin{pmatrix} 0 & -\boldsymbol{S}_2\\ \boldsymbol{S}_1 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{e}\\ \boldsymbol{h}\end{pmatrix} = \lambda^{-1}\begin{pmatrix}\boldsymbol{e}\\ \boldsymbol{h}\end{pmatrix}. \tag{4}$$

Moreover, if $\boldsymbol{e}, \boldsymbol{h} \in \boldsymbol{L}^2(\Omega)$ satisfy (4), then they are eigenfunctions of the Maxwell system. $\boldsymbol{B}$ is a *compact, skew-Hermitian* operator on $\boldsymbol{L}^2_\varepsilon(\Omega)\times\boldsymbol{L}^2_\mu(\Omega)$. Therefore, $-\boldsymbol{B}^2$ is *positive, semidefinite*, and *Hermitian*. Its eigenpairs, and therefore the eigenpairs of (1), can be obtained by solving

$$\boldsymbol{S}_2\boldsymbol{S}_1\phi = \tau^2\phi. \tag{5}$$

3 Least-Squares Discretization

Assume mesh partitioning of Ω. Define discrete solution space: $X_h \subset \boldsymbol{L}^2(\Omega)$, discrete test space: $\boldsymbol{Y}_{h,k} \subset \boldsymbol{Y}_h$. In the simplest case, X_h – piecewise constants and $\boldsymbol{Y}_{h,k}$ – piecewise linear + face bubble functions in each component. The bubbles are required in our method to insure the solvability of the approximation to (3).

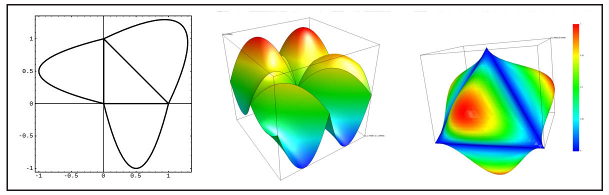


Figure 1. *Face bubble functions in two and three dimensions.*

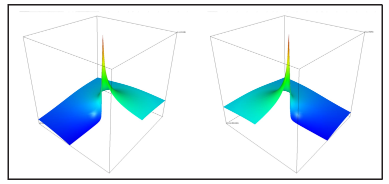
For each equation in (3), we apply a least-squares discretization, based in the $\boldsymbol{H}^{-1}(\Omega)$ inner product. This leads to an algebraic system with a symmetric and positive definite matrix. The matrix is full, but its action can be easily computed by inverting the standard piecewise linear stiffness matrix. Inversion can be replaced by a preconditioner, e.g. few sweeps of multigrid on each application of the matrix. Here are few advantages of the resulting algorithm:

- We have optimal order error estimates with minimal regularity assumptions.
- Our method is based in $\boldsymbol{L}^2(\Omega)$, so the algebraic system is symmetric, positive definite and well-conditioned. Thus the action of $\boldsymbol{S}_{h,k} : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{X}_h$, the discrete analogs of $\boldsymbol{S}_k$, are relatively easy to compute.
- We were able to proof that the operators $\boldsymbol{S}_{h,k}$ converge in operator norm to $\boldsymbol{S}_k$.
- This implies that the eigenvalues of the discrete operator $\boldsymbol{S}_{h,2}\boldsymbol{S}_{h,1}$ will converge to the inverses of the Maxwell eigenvalues, and there will be no spurious eigenvalues as in some other methods.

4 Magnetostatics

Here we demonstrate the computational behavior of the proposed method on few examples in the case of magnetic fields produced by steady currents. The iteration stopping criterion in PCG is reduction of the initial residual by 6 orders of magnitude. We expect number of iterations bounded independently of the number of unknowns.

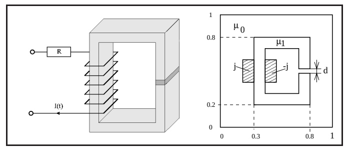
The first problem is posed on a L-shaped domain. The solution is only in $H^{1+s}(\Omega)$ for $s < \frac{2}{3}$. The components of the magnetic field are shown on the next figure. One can clearly see the singularity at the origin. Numerical experiments confirm the expected convergence rate ($2^{2/3} = 1.5874...$).



h	$\lVert e\rVert_0$	$ratio$	n_{it}	N	$time$
0.1767	0.22271		12	512	0.03
0.0883	0.14253	1.5588	13	2048	0.11
0.0441	0.09072	1.5712	13	8192	0.59
0.0220	0.05749	1.5779	13	32768	2.51
0.0110	0.03635	1.5817	13	131072	12.8

Figure 2. *Magnetostatic problem in a L-shaped domain.*

The next problem is a cross-section of a magnet with small air gap. In this case μ has a jump of 4 orders of magnitude ($\mu_0 = 1$, $\mu_1 = 10^4$). The source current is zero except for the shaded parts. Note the first order convergence, and that using multigrid instead of direct solver leads to a reduction of more than 16 in the computational time.



h	$\lVert e\rVert_0$	n_{it}	N	$time(mg)$	$time(exact)$
0.0316	0.3162	9	152	0.02	0.01
0.0158	0.1581	16	608	0.05	0.17
0.0079	0.0790	19	2432	0.19	2.59
0.0039	0.0395	20	9728	1.11	14.3
0.0019	0.0197	21	38912	5.33	85.9

Figure 3. *Cross-section of a three dimensional magnet.*

Next example is a three dimensional transformer. The right hand side, specifies rotational currents in the three coils, μ is four orders of magnitude larger in the iron core.

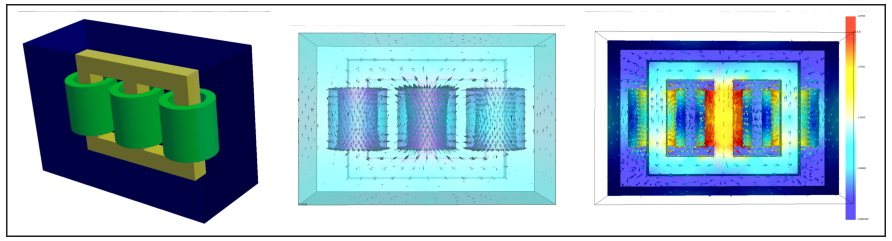


Figure 4. *The transformer problem.*

5 The Eigenvalue Problem

The eigenvalue problem has many applications, one of which is the design of linear accelerators, where the eigenmodes of the structure correspond to the resonating frequencies and thus are of critical importance.

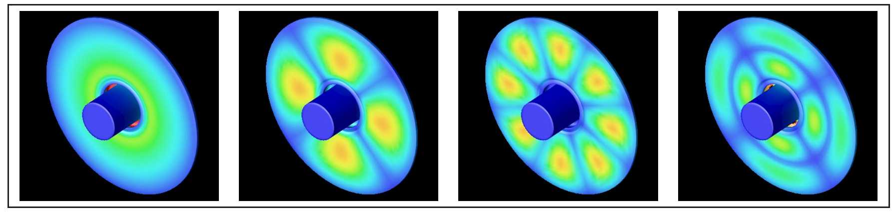
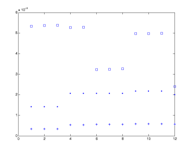


Figure 5. *Example of an accelerator induction cell (about 1m in diameter). Some of the lowest eigenmodes (1st, 5th, 12th and 20th) are shown. Courtesy of EMSolve's project.*

Here we report our eigenvalue computations for model cases in two and three dimensions.



h	$\lvert\lambda_{max} - \lambda_{h,max}\rvert$	n_{it}	N	$time(mg)$
0.125	0.0014714	13	384	0.67
0.0625	0.0004549	11	1536	3.72
0.03125	0.0001069	9	6144	22.6
0.015625	0.0000267	8	24576	250
0.0078125	0.0000067	7	98304	2577

Figure 6. *Eigenvalue computation with $\varepsilon = \mu = 1$ for the maximal 12 eigenvalues in the unit cube (left) and the maximal eigenvalue in the unit square (right).*

6 Current Research

Our current work is on eigenvalue computations for practical problems which are of interest for the lab. This involves computations of large three-dimensional problems which necessarily are done in parallel. The parallel implementation is based on data structures and preconditioners from the *hypre* package.

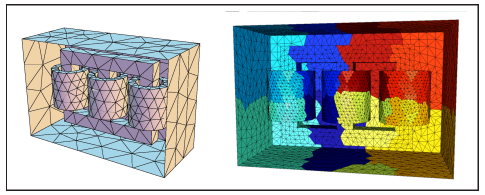


Figure 7. *Initial mesh for the transformer which is split and refined on 8 processors.*