NUMERICAL ANALYSIS QUALIFIER

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Do all of the following five problems.

Problem 1. Let A be a real, symmetric, nonsingular matrix of dimension n (A not positive definite). Let $(\cdot, \cdot)$ denote the dot inner product on R^n.

(a) Derive a steepest descent iteration for solving

$$Ax = b \tag{1.1}$$

of the form

$$x_n = x_{n-1} + \alpha_n r_{n-1}, \qquad r_{n-1} = b - Ax_{n-1}$$

which minimizes the error $e_n = x - x_n$ in the norm

$$\|e_n\|_{A^2} \equiv (Ae_n, Ae_n).$$

(b) Given an initial iterate x_0, set $r_0 = b - Ax_0$ and let

$$K_n = \operatorname*{span}_{i=0}^{n-1}\{A^i r_0\}$$

be the Krylov space. Let $x_n = x_0 + \chi$ where $\chi \in K_n$ is such that $e_n = x - x_n$ satisfies

$$\|e_n\|_{A^2} = \min_{\theta \in K_n} \|x - (x_0 + \theta)\|_{A^2}. \tag{1.2}$$

Show that χ (and hence x_n) is uniquely defined. This method can be implemented using a conjugate gradient type algorithm.

(c) Derive an estimate for the rate of iterative convergence for the method satisfying (1.2) in terms of the largest and smallest eigenvalue of the matrix A^2. (Hint: Start by showing that this method with $n = 2l$ is at least as good as l steps of the conjugate gradient method applied to $A^2 x = Ab$ with the same initial iterate.)

Problem 2. Consider Simpson's rule

$$I(f) = \frac{1}{3}(f(-1) + 4f(0) + f(1))$$

for approximating the integral

$$\int_{-1}^{1} f(x)\, dx.$$

(a) Show that $I(f)$ is exact for quadratics.
(b) Compute the Peano Kernel $K_2(t)$ for the error.
(c) Use the Peano Kernel Theorem to show that for $f \in C^3[-1, 1]$,

$$\left|\int_{-1}^{1} f(x)\, dx - I(f)\right| \le \frac{1}{36} \max_{x \in [-1,1]} |f'''(x)|. \tag{2.1}$$

Problem 3. Consider the complex valued boundary value problem

$$u - \Delta u + i\omega u = f \quad \text{in} \quad \Omega$$

$$iu + \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega.$$

Here i denotes the square root of minus one, n is the outward normal on $\partial\Omega$ and ω is a real number.

(a) Rewrite the above boundary value problem as a system of PDE's and boundary conditions involving the real and imaginary parts of u, (u_r and u_i, respectively).
(b) Derive a weak formulation of the above problem which gives rise to a coercive bilinear form on the space $H^1(\Omega)^2$.
(c) Show that the form of Part b is coercive.

Problem 4. Let Ω be a polygonal domain in R^2 and consider the problem: Find $u \in V \equiv H^1(\Omega)$ satisfying

$$a(u, v) = f(v) \ \forall \ v \in V,$$

where

$$a(u, v) \equiv \int_\Omega (\nabla u \cdot \nabla v + uv) \ dx\, dy, \ \ f(v) \equiv \int_\Omega fv \ dx\, dy, \ \ f \in L^2(\Omega).$$

Denote $\mathcal{T} = \cup K_i$ to be an admissible triangulation of Ω and P_2 to be the set of polynomials in x and y of degree 2. For each triangle, consider the degrees of freedom for P_2 corresponding to the values at the vertexes and values of the normal derivatives at the centers of the edges.

(a) Show that a function in P_2 which vanishes at the above degrees of freedom has zero gradient at the centers of the edges.
(b) Use Part a above to show that the above degrees of freedom form a unisolvent set for P_2.
(c) Prove or disprove: The piecewise quadratic space defined with respect to $\mathcal{T}$ and these degrees of freedom a subset of V.

Problem 5. Given u_i^0, u_i^1, for $i \in Z$ and $k, h, b > 0$ consider the Du Fort-Frankel scheme:

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + b\frac{v_m^{n+1} + v_m^{n-1} - v_{m-1}^n - v_{m+1}^n}{h^2} = f_m^n, \quad m \in Z, \ n = 1, 2, \ldots.$$

With u_i^0, u_i^1 and f_m^n appropriately chosen, the discrete solution approximates the solution ($v_m^n \approx u(mh, nk)$) of the parabolic initial value problem

$$\frac{\partial u}{\partial t} - b\frac{\partial^2 u}{\partial x^2} = f, \quad x \in R, \ t > 0$$

$$u(x, 0) = u_0(x), \quad x \in R.$$

(a) Give a bound for the local truncation error associated with the above scheme.
(b) Using Fourier mode analysis, determine the stability properties of the above scheme.