NUMERICAL ANALYSIS QUALIFIER

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Do all of the following five problems.

Problem 1. Let A be a symmetric and positive definite matrix in $\mathbb{R}^{n\times n}$. Consider the linear system $Ax = b$ and the following two step iteration: Given $x^0 \in \mathbb{R}^n$, define for $m = 0, 2, \ldots$,

$$x^{m+1} = (I - \tau_1 A)x^m + \tau_1 b,$$
$$x^{m+2} = (I - \tau_2 A)x^{m+1} + \tau_2 b.$$

Here I denotes the identity matrix and τ_1 and τ_2 are iteration parameters.

(a) Given only the lower bound $0 < \lambda$ and the upper bound Λ for the eigenvalues of A find the iteration parameters τ_1 and τ_2 (in terms of λ and Λ) which give rise to the optimal convergence rate when measured in the Euclidean norm.
(b) Compute the convergence rate corresponding to these parameters.
(c) Find lower and upper bounds for the eigenvalues in terms of n for the matrix A with entries $(-1, 2, -1)$ (on the lower co-diagonal, diagonal and upper co-diagonal, respectively).

Problem 2. Consider the quadrature

$$I(f) = a_0 f(1/2) + a_1 f(t) \approx \int_{-1}^{1} f(x)\,dx,$$

where t is in $[-1, 1]$.

(a) Find a_0, a_1 and t such that the quadrature rule is exact for polynomials of highest degree.
(b) Explicitly compute the Peano kernel for the quadrature error in part (a) and give the Peano Kernel theorem representation of the error (note that you do not need to evaluate or bound the integral in the representation formula).

Problem 3. Consider the following boundary value problem: Find $u(x) \in C(0, 1)$ such that

$$\begin{aligned} -u'' &= f(x), \qquad x \in I_1 := (0, 1/2) \cup (1/2, 1), \\ u'(\tfrac{1}{2}+) - u'(\tfrac{1}{2}-) &= 5u(\tfrac{1}{2}), \\ u'(0) = u'(1) &= 0. \end{aligned}$$

Here $f \in L^2(0, 1)$ and $v(\xi-)$ and $v(\xi+)$ are the limit values of the function v when x goes to ξ from the left and right, respectively.

(a) Derive the weak formulation of this problem using an appropriate Hilbert space V.
(b) Show that the bilinear form of part (a) is coercive and continuous on V.

(c) Consider the finite element method for this problem based on linear finite elements over a uniform partition of $[0,1]$ with mesh-size $h = 1/m$. Show that the Galerkin approximation is first order accurate in the H^1-norm if the mesh is aligned with the point $x = 0.5$. (You may use the fact that u satisfies the *a priori* estimate

$$||u''||^2_{L^2(I_1)} + ||u''||^2_{L^2(I_2)} = ||f||^2_{L^2(0,1)}.$$

This follows immediately from the equation. Note, however, that u is generally not in $H^2(0,1)$.)

Problem 4. (a) Consider the tensor product space $Q_2 = \{\sum_{i,j=0}^{2} c_{i,j} x^i y^j\}$ and the the reference square $\hat{\tau} \equiv [0,1]^2$ with nodes as illustrated below

$$\begin{array}{ccccc} v_1 & \text{——} & v_5 & \text{——} & v_2 \\ | & & | & & | \\ v_8 & \text{——} & v_9 & \text{——} & v_6 \\ | & & | & & | \\ v_4 & \text{——} & v_7 & \text{——} & v_3. \end{array}$$

Show that these nodes form a unisolvent set for Q_2.

(b) Let $\{\phi_i\}$, $i = 1, \ldots, 9$ be the nodal basis functions associated with the above nodes. Compute ϕ_4 explicitly.

(c) Consider the reduced space

$$Q_2' = \left\{ p \in Q_2 \ : \ 4p(v_9) + \sum_{i=1}^{4} p(v_i) - 2 \sum_{i=5}^{9} p(v_i) = 0 \right\}.$$

Show that Q_2 contains P_2, the space of polynomials in x and y of total degree at most 2.

(d) Let Ω be a rectangular domain and $\Omega = \cup \overline{\tau}_i$ be an admissible partitioning of Ω into smaller rectangular elements. Let S_h be the space of piecewise (with respect to this partitioning) Q_2' functions based on the reference nodes $\{v_i\}$, $i = 1, \ldots, 8$ (this set of nodes forms a unisolvent set for Q_2'). Show that S_h is contained in $H^1(\Omega)$

Problem 5. Consider the parabolic problem

$$\begin{aligned} u_t - u_{xx} &= f, \quad \text{for} \quad (x,t) \in (0,1) \times (0,T], \\ u(0,t) = u(1,t) &= 0, \quad \text{for} \quad t \in (0,T], \\ u(x,0) &= u_0(x), \quad \text{for} \quad x \in [0,1]. \end{aligned}$$

Here f and u_0 are given sufficiently smooth functions and $T > 0$.

(a) Let S_h be a finite element approximation subspace in $H_0^1(\Omega)$ and $U^0 \in S_h$ be an approximation to u_0. For θ in $[0,1]$ consider the fully discrete scheme: $U^n \in S_h$

satisfies

$$\left(\frac{U^n - U^{n-1}}{k}, \phi\right) + D(\theta U^n + (1-\theta)U^{n-1}, \phi) = (f(t_{n+\theta}), \phi), \tag{5.1}$$

for all $\phi \in S_h$. Here

$$(v, w) \equiv \int_0^1 vw\, dx, \qquad D(v, w) \equiv (v_x, w_x),$$

k is the time step size, $t_\alpha = \alpha k$ and U^n approximates the solution $u(\cdot, t_n)$. Show that (5.1) is unconditionally stable when θ is in $[1/2, 1]$.

(b) Define the analogous finite difference method using the 3-point stencil as an approximation to u_{xx}.

(c) Using Fourier mode analysis, derive a CFL condition (depending on θ) for the finite difference approximation of part (b) which guarantees stability when θ is in $[0, 1/2)$.