Theorem 1. *Let $B \subset M_k(\mathbb{C})$ be a semi-simple algebra. If B has property P_1, then $dimB \leq k$. Furthermore, if $dimB = k$, then B is a maximal P_1 algebra.*

Our primary goal is to prove this theorem. A maximal P_1 algebra is an algebra B with property P_1 such that any algebra $A, B \subsetneq A$, A does not have property P_1. To prove this theorem, we will need a few lemmas first to assist us.

Lemma 1. *Let $B \subset M_k(\mathbb{C})$ be a semi-simple algebra. If B has property P_1, then $dimB \leq k$.*

Proof. We will use induction on k. The case $k = 1$ is clear. Suppose this is true for $k \leq n$ and let $B \subset M_{n+1}(\mathbb{C})$ be a semi-simple algebra. We need to show $dimB \leq n + 1$. Suppose B has a non-trivial central projection, $p, 0 < p < 1$. Then, $B = pBp \oplus (1 - p)B(1 - p)$. From this, we can see $pBp \subset B(pH)$ and $(1 - p)B(1 - p) \subset B((1 - p)H$ are both semi-simple algebras with property P_1, as proven earlier. By the assumption of induction $dimpBp \leq dim(pH)$ and $dim(1 - p)B(1 - p) \leq dim(1 - p)H$. Therefore, $dimB = dim(pBp) + dim((1 - p)B(1 - p)) \leq dimpH + dim(1 - p)H = dimH = n + 1$.

Now, let's assume B does not have a nontrivial central projection. Then, $B = M_r(\mathbb{C}) \subset M_{n+1}(\mathbb{C})$. Since B has P_1, $r^2 \leq n + 1$, so $r \leq n + 1$.

□

Lemma 2. *Let $B \subset M_4(\mathbb{C})$. If $B = M_2(C)$, then B is a maximal P_1 algebra.*

Lemma 3. *Suppose $0 \neq a \in M_n(\mathbb{C})$. Then, for any $a \in A$, there exists finite elements $b_1, b_2....b_k, c_1, c_2, ...c_k$, such that $\sum_{i=1}^{k} b_i a c_i = I_n$.*

Proof. Note that $M_n(\mathbb{C})aM_n(\mathbb{C})$ is a two sided ideal of $M_n(\mathbb{C})$ and $M_n(\mathbb{C})aM_n(\mathbb{C}) \neq 0$. So, $M_n(\mathbb{C})aM_n(\mathbb{C}) = M_n(\mathbb{C})$, this implies the lemma. □

The following well known lemma will be very helpful.

Lemma 4. *There are finitely many unitary matrices, $u_1, u_2, ...u_k \in M_n(\mathbb{C})$, such that $\frac{1}{k}\sum_{i=1}^{k} u_i a u_i^* = \frac{Tr(a)}{n} I_n$ for all $a \in M_n(\mathbb{C})$.*

Lemma 5. *Let $b \subset M_{n^2}(\mathbb{C})$. If $B = M_n(\mathbb{C})$, then B is a maximal P_1 algebral.*

We may write $M_{n^2}(\mathbb{C}) = M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ and assume $B = M_n(\mathbb{C}) \otimes I_n$. Since B has a separating vector, B has property P_1.

Now, assume $B \subsetneq R \subseteq M_{n^2}(\mathbb{C})$ and R is a P_1 algebra. We can write $R = R_1 + J$, such that $B \subseteq R_1$, where R_1 is the semi-simple part and J is the radical of R. Since R has P_1, R_1 has P_1, and by Lemma 1, $dim R_1 \leq n^2$. Since $dim B = n^2$, we have $R_1 = B$.

Suppose $0 \neq x = (x_{ij})_{1 \leq i,j \leq n} \in J$ with respect to the matrix units $I_n \otimes M_n(\mathbb{C})$. Note that with respect to the matrix units of $I_n \otimes M_n(\mathbb{C})$, each element of $B = M_n(\mathbb{C}) \otimes I_n$ has the following form $\begin{pmatrix} a & . & . & 0 \\ 0 & a & .. & 0 \\ . & & & \\ . & & & \\ 0 & 0 & ... & a \end{pmatrix}), a \in M_n(\mathbb{C})$. Without loss of generality, let's assume $x_{11} \neq 0$.

By Lemma 3, there exists a finite elemtns $b_1, ...b_k, c_1, ...c_k \in M_n(\mathbb{C})$, such that

$$\sum_{i=1}^{k} b_i x_{11} c_i = I_n \ . \tag{1}$$

Let $y = (y_{ij})_{1 \leq i,j \leq n} = \sum_{i=1}^{k} (b_i \otimes I_n) X (c_i \otimes I_n) \in J$. By (1), we have $y_{11} = I_n$. Next, we can choose unitary matrices $u_1, ...u_k$ as in Lemma 4. Let $z = (z_{ij}) = \sum_{i=1}^{k} (u_i \otimes I_n) Y (u_i^* \otimes I_n) \in J$. Then, $z_{11} = I_n$ and $z_{ij} = \lambda_{ij} I_n$ for some $\lambda_{ij} \in \mathbb{C}, 1 \leq i,j \leq n$. So, $Z \in I_n \otimes M_n(\mathbb{C})$.

Since $z \in J$, $z^n = 0$, as elements in the radical are nilpotent. By the Jordan Canonical theorem, there exists an invertible matrix $w \in I_n \otimes M_n(\mathbb{C})$ such that $0 \neq wzw^{-1} = \oplus_{i=1}^{k} z_i \in I_n \otimes M_n(\mathbb{C})$ and each z_i is a Jordan block with diagonal 0. By replacing R with wRw^{-1}, we may assume $0 \neq z = \oplus_{i=1}^{k} z_i \in I_n \otimes M_n(\mathbb{C})$.

Suppose $r = max\{rankz_i : 1 \leq i, \leq k\}$. We may assume $rankz_1 = ... = rankz_s = r$ and rank $z_i < r$ for all $s < i \leq k$. Then $z^{r-1} = (\oplus_{i=1}^{s} z^{r-1}) \oplus 0$. Note that $z^{r-1} = \begin{pmatrix} 0 & ... & 0 & 1 \\ 0 & ... & 0 & 0 \\ \vdots \\ 0 & ... & 0 & 0 \end{pmatrix}$. We may assume R is the algebra generated by $M_n(\mathbb{C}) \otimes I_n$ and $I_n \otimes z^{r-1}$.

Without loss of generality, we assume $r = 2$, and hence $s = \frac{n}{2}$. The general case can be proved similarly. Let $t = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right), a, b \in M_n(\mathbb{C})$. Then, $R = \left\{ \begin{pmatrix} t & 0 & ...0 \\ 0 & t & 0... \; 0 \\ \vdots \\ 0 & 0... & t \end{pmatrix} \right\}$. Let $t_{i\perp} = \left(\begin{smallmatrix} x_{2i-1} & * \\ y_i & x_{2i} \end{smallmatrix}\right)$. Then, simple computations show that $R_\perp = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_{\frac{n}{2}} \end{pmatrix} \right\}$.Let $m = \left(\begin{smallmatrix} 0 & 0 \\ I_n & 0 \end{smallmatrix}\right)$). Since this has P_1, we should be able to write $\begin{pmatrix} m & & & \\ & m & & \\ & & \ddots & \\ & & & m \end{pmatrix}$ plus an element of the preannihilator as a rank-1 matrix. However, if this is so, then we know $1 + y_1, 1 + y_2 ... 1 + y_s$ are all rank-1. However, summing all of these gives $I_n + y_1 + I_n + y_2 + ...I_n + y_s = s * I_n$ which is rank at most $s = \frac{n}{2} < n$. This is a contradiction.

Lemma 6. *Suppose* $\lambda \neq 0 \in \mathbb{C}$ *and*$y_1, y_2, ..., y_2 \in M_n(\mathbb{C})$ *such that* $y_1 + y_2 + ... + y_n = 0$. *Suppose* $\eta_1, \eta_2, ..., \eta_n \in \mathbb{C}^n$ *are linearly dependent. Let*

$$t = \begin{pmatrix} \lambda & * & . & . & . & * \\ \eta_1 & I_n + y_1 & * & . & . & * \\ \eta_2 & * & I_n + y_2 & * & ... & * \\ . \\ . \\ . \\ \eta_n & * & ... & * & * & I_n + y_n \end{pmatrix}.$$ *This matrix has rank ¿ 1.*

Proof. Note first that each η_i block is an $n \times 1$ column vector. Since we are saying they are linearly dependent, then we know that there are k vectors in the set $\{\eta_i\}_{i=1}^{k}$ that are independent. Without loss of generality, assume that the first k vectors are the linearly independent ones. Then, for any $j > k$, η_j can be written as a linear combination of the first k elements. Another way of viewing this is saying that if we look at the matrix $[\eta_1\eta_2...\eta_n]$, for any $j > k$, the j-th row can be written as a linear combination of the first k rows. So, in our matrix t, let's assume it has rank one. On each η_i's j-th row, we can row reduce them to zero. To maintain rank-1, since we have the nonzero-entry in the top left, we have to have the entire row containing a j-th entry has to be zero. Doing this row reduction changes our y_i to a $y_i^{'}$ such that $I_n + y_i^{'}$ has zero entries along it's row that it shares with the j-th entries of each η_i. However, we still maintain the condition that $\sum_{i=1}^{n} y_i^{'} = 0$. These rows that contain these j row entries occur in the $k * j + 1$ row where $1 \leq k \leq n$. So, since we know all these rows have to be zero, we know something about the $1 + y_i$'s $i * j + 1$ entry. We know it has to be zero now. So, we can sum up each of those new 0 entries from each $1 + y_i^{'}$. Doing this sum only over the position that it shares with the j-th row of each η_i gives $0 = \sum_{i=1}^{n} 1 + y_i^{'} = \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} y_i^{'} = \sum_{i=1}^{n} 1 = n$. However, that gives us $n = 0$, which is impossible, hence contradicting our claim that this is rank-1.

$\square$

Lemma 7. *Let $B =\subset M_5(\mathbb{C}) = B(H)$ such that $dim H = 5$ and* $B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \lambda \in \mathbb{C}, a \in M_2(\mathbb{C}) \right\}$ *Then, B is a maximal P_1 algebra.*

Proof. Since B has a separating vector, B has property P_1. Suppose $B \subset R \subseteq M_5(\mathbb{C})$ and R is a $P_1 algebra$. We can write $R = R_1 + J$, such that $B \subseteq R_1$, where R_1 is the semi-simple part and J is the radical part. By Lemma 1, $B = R_1$. Let $0 \neq X \in J$ and let $p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0_2 \\ 0 & 0_2 & I_2 \end{pmatrix}$. Then $qBq \subseteq qRq \subset B(PH) = M_4(\mathbb{C})$. By Lemma 3, $qBq = qRq$. This implies we may assume $0 \neq x = \begin{pmatrix} 0 & \xi^T & \eta^T \\ 0 & 0_2 & 0_2 \\ 0 & 0_2 & 0_2 \end{pmatrix}$, where $\xi, \eta \in \mathbb{C}^2$.

Case 1: ξ and η are linearly independent. Then, $x \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in R$. Since ξ and η are linearly independent, and $a \in M_2(\mathbb{C})$ is arbitrary, this implies that $R = \left\{ \begin{pmatrix} \lambda & \xi^T & \eta^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \lambda \in \mathbb{C}, \xi, \eta \in \mathbb{C}^2, a \in M_2(\mathbb{C}) \right\}$. Simple computations show that $R_\perp = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & y_1 & * \\ 0 & * & y_2 \end{pmatrix} y_1, y_2 \in M_2(\mathbb{C}), y_1 + y_2 = 0 \right\}$ Since we assume R has property P_1, $I_5 + R_\perp$ is rank-1 for some matrix in $R_\perp$. This gives us a matrix of the form $R_\perp = \begin{pmatrix} 1 & * & * \\ 0 & y_1+I_2 & * \\ 0 & * & y_2+I_2 \end{pmatrix}$. However, this contradicts Lemma 7.

Case 2: ξ and η are linearly dependent. Without loss of generality, assume $\eta = t\xi$, so $x = \begin{pmatrix} 0 & \xi^T & t\xi^T \\ 0 & 0_2 & 0_2 \\ 0 & 0_2 & 0_2 \end{pmatrix}$ and $x \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & \xi^T & t\xi^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$. Since $\xi \neq 0$, and $a \in M_2(\mathbb{C})$ is arbitrary, this implies that $R = \left\{ \begin{pmatrix} \lambda & \xi^T & t\xi^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \lambda \in \mathbb{C}, \xi \in \mathbb{C}^2, a \in M_2(\mathbb{C}) \right\}$. Simple computations show that

$$R_\perp = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \eta_1 & y_1 & * \\ \eta_2 & * & y_2 \end{pmatrix} y_1, y_2 \in M_2(\mathbb{C}), y_1 + y_2 = 0, \eta_1, \eta_2 \in \mathbb{C}^2, \eta_1 + \eta_2 = 0 \right\} \tag{2}$$

If this space has P_1, then $I_5 + R_\perp$ should be rank-1 for some element of $R_\perp$. However, this gives us matrices of the form $R_\perp = \begin{pmatrix} 1 & * & * \\ \eta_1 & y_1+I_2 & * \\ \eta_2 & * & y_2+I_2 \end{pmatrix}$, which contradicts lemma 7. $\square$

Lemma 8. *Suppose $z_{ij} \subseteq M_{sr}(\mathbb{C})$ and $\{c_{ji}\} \subseteq M_{rs}(\mathbb{C})$ such that* $\sum_{i=1}^s \sum_{j=1}^r z_{ij} a c_{ji} b =$

0 $\quad, \forall a \in M_r(\mathbb{C}), b \in M_s(\mathbb{C})$. *If $c_{ji} \neq 0$ for some $1 \leq i \leq s, 1 \leq j \leq r$, then z_{ij} are linearly dependent.*

Proof. We may assume $c_{11} \neq 0$ and the $(1,1)$ entry of c_{11} is not zero. Replace c_{ji} by $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & 0 \end{pmatrix} c_{ji} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & 0 \end{pmatrix}$, we may assume $c_{ji} = \lambda_{ij} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & 0 \end{pmatrix}, \lambda_{11} = 1$.

Let x_{ij}^k be the k-th column of z_{ij}.Note that $z_{ij} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & 0 \end{pmatrix} = x_{ij}^1$. Then, $\sum_{i=1}^s \sum_{j=1}^r z_{ij} c_{ji} = 0$ implies $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} z_{ij} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & 0 \end{pmatrix} = 0$ which implies $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} x_{ij}^1 = 0$

Similarly, we can use $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & 0 \end{pmatrix} c_{ji} = \lambda_{ij} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & 0 \end{pmatrix}$ show $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} x_{ij}^2 = 0$. Proceeding similarly, we obtain $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} x_{ij}^k = 0$ for all $1 \leq k \leq r$. Therefore, $\sum_{i=1}^s \sum_{j=1}^r \lambda_{ij} z_{ji} = 0$ which shows the z_{ji} are linearly dependent.

□

Lemma 9. *Let $B \subseteq M_{r^2+s^2}(\mathbb{C}) = B(H)$ such that $dimH = (r^2+s^2)^2$ and $B = \{a^{(r)} \oplus b^{(s)} : a \in M_r(\mathbb{C}), b \in M_s(\mathbb{C})\}$. Then, B is a maximal P_1 algebra.*

Proof. Since B has a separating vector, B has property P_1. Suppose $B \subsetneq R \subseteq M_{r^2+s^2}(\mathbb{C})$ such that R has P_1. Write $R = R_1 + J$ such that $B \subseteq R_1$, where R_1 is the semi-simple part and J is the radical part. . By Lemma 1, $B = R_1$. Let $0 \neq X \in J$ and let $p = I_r^{(r)} \oplus 0$ and $q = p = I_s(s) \oplus 0$. Then, $pBp \subseteq pRp \subseteq B(pH)$ and pRp has property P_1. By Lemma 5, $pRp = pBp$. Similarly, $qRq = qBq$. This implies we may assume $0 \neq x = \begin{pmatrix} 0_r^{(r)} & C \\ 0 & 0_s^{(s)} \end{pmatrix}, C \neq 0$. If $Z \in R_\perp$ such that

$$Z = \begin{pmatrix} x_1 & * & ... & * & * & * & & & \\ * & x_2 & * & .... & * & & & & \\ & & \ddots & & & & & & \\ * & * & * & x_r & \cdots & & & & \\ z_{11} & z_{12} & z_{13} & \cdots z_{1r} & y_1 & * & \cdots & * & \\ z_{21} & z_{22} & z_{23} & \cdots z_{2r} & * & y_2 & * & \cdots & * \\ \vdots & & & & & & & & \\ z_{s1} & z_{s2} & z_{s3} & \cdots z_{sr} & * & \cdots & *y_s & & \end{pmatrix}$$

Then $x_1+x_2+...+x_r = 0_r$ and $y_1+y_2+...y_s = 0_s$. Note that $x(a^{(r)} \oplus b^{(s)} = \begin{pmatrix} 0_r^{(r)} & cb^{(s)} \\ 0 & 0_s^{(s)} \end{pmatrix}$. Write $c = (c_{ij})_{1\le i\le r, 1\le j\le s}$. Therefore, we have

$$Tr\left(\begin{pmatrix} z_{11} & \dots & z_1r \\ \vdots & & \\ z_{s1} & \dots & z_{sr} \end{pmatrix}\begin{pmatrix} c_{11} & \dots & c_{1s} \\ \vdots & & \\ c_{r1} & \dots & c_{rs} \end{pmatrix}\begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix}\right) = 0$$

Simple computation shows that $Tr\left(\sum_{i=1}^{s}\sum_{j=1}^{r} z_{ij}c_{ji}b\right) = 0$. Since $b \in M_s(\mathbb{C})$ is arbitrary $\left(\sum_{i=1}^{s}\sum_{j=1}^{r} z_{ij}c_{ji}\right) = 0$.

Note that

$$(a^{(r)} \oplus 0)x(0 \oplus b^{(s)}) = \begin{pmatrix} 0_r^{(r)} & a^{(r)}cb^{(s)} \\ 0 & 0_s^{(s)} \end{pmatrix} = \begin{pmatrix} 0_r^{(r)} & (ac_{ij}b)_{1\le i\le r, 1\le j\le s} \\ 0 & 0_s^{(s)} \end{pmatrix} \tag{3}$$

So, we have $\sum_{i=1}^{s}\sum_{j=1}^{r} z_{ij}ac_{ji}b = 0, \quad \forall a \in M_r(\mathbb{C}), b \in M_s(\mathbb{C})$. By Lemma 8, this implies that z_{ij} are linearly dependent.

Suppose $I_{r^2+s^2} + z$ is rank 1 for some $z \in R_\perp$. Then $(z_{ij})_{1\le i\le r, 1\le j\le s}$ are rank1 matrices. So there are $\xi_1...\xi_s \in \mathbb{C}^s, \eta_1...\eta_r \in \mathbb{C}^r$ such that $z_{ij} = \xi_i \otimes \eta_j$. Since $\{z_{ij}\}$ are linearly dependent, either $\{\xi_i\}$ are linearly dependent or $\{\eta_j\}$ are linearly dependent. Without loss of generality, assume $\{\xi_i\}$ are linearly dependent. Now, $I_{r^2+s^2}+z$ is a matrix of the form $\begin{pmatrix} I_r+x_1 & & & & & \\ & & \ddots & & & \\ \xi_1\otimes\eta_1 & \dots & \xi_n\otimes\eta_1 & I_s+y_1 & & \\ \vdots & & & & & \\ \xi_s\otimes\eta_1 & \dots & \xi_s\otimes\eta_r & * & \dots & I_s+y_s \end{pmatrix}$

Since $x_1+...+x_r=0$, one entry of I_r+x_i is not zero for some $1\le i\le r$. We may assume the $(1,1)$ entry of I_r+x_1 is not zero. Let $\eta_1=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_r\end{pmatrix}$. Then the matrix

$$\begin{pmatrix} I_r+x_1 & & & & & \\ & & \ddots & & & \\ \alpha_1\xi_1 & \dots & I_s+y_1 & & & \\ \vdots & & & & & \\ \alpha_1\xi_s & \dots & & * & \dots & I_s+y_s \end{pmatrix}$$

By lemma 6, this matrix has rank ≥ 2. This contradicts our assumption. □

We are now ready to prove Theorem 1.

Proof. By Lemma 1, if B has P_1, then $dimB\le k$. Assume B has property P_1, and $dimB=k$. We claim $B=\oplus_{i=1}^r M_{(n_i)}^{n_i}(\mathbb{C}), k=\sum_{i=1}^r n_i^2$. We will proceed by induction on k. If $k=1$, this is clear. Assume our claim is true for $k\le n$. Let $B\subseteq M_{n+1}(\mathbb{C})$ be a semi-simple P_1 algebra and $dim(B)=n+1$Suppose B has non trivial central projection $p, 0<p<1$. Then, $B=pBp\oplus(1-p)B(1-p)$. From this we can say $pBp\subseteq B(pH)$ and $(1-p)B(1-p)\subseteq B((1-p)H)$ are both semi-simple with property P_1. By Lemma 1 $dim(pBp)=dim(pH)$ and $dim((1-p)B(1-p))=dim((1-p)H)$. By induction, $pBp=\oplus_{i=1}^r M_{n_i}^{n_i}(\mathbb{C})$ and $(1-p)B(1-p)=\oplus M_{n_i}^{n_i}(\mathbb{C})$.

Suppose B does not have a nontrivial central projection. Then $B=M_r(\mathbb{C})\subseteq M_{n+1}(\mathbb{C})$ and $dimB=r^2=n+1$, so $B=M_r(\mathbb{C})^{(r)}$.

Suppose $B\subsetneq R\subseteq M_k(\mathbb{C})\in B(H)$ such that R has property P_1. Let

$B = R_1$. Let p_i be the projection of B that corresponds to the $M_{n_i}^{(n_i)}$. Let $0 \neq x \in J$. Then, we have $p_i B p_i \subseteq p_i R p_i \subseteq B(p_i H)$ and $p_i R p_i$ has property P_1. By Lemma 5 $p_i R p_i = p_i B p_i$, this implies we may assume $0 \neq x =$

$$\begin{pmatrix} 0_{n_1}^{(n_1)} & * & x_{12} & \dots & x_{1n_i} \\ 0 & 0_{n_2}^{(n_2)} & & & \\ & & \ddots & & \vdots \\ 0 & \dots & & & 0_{n_i}^{(n_i)} \end{pmatrix}$$

We now assume $x_{12} \neq 0$. Then $(p_1 + p_2)x(p_1 + p_2) \subsetneq (p_1 + p_2)R(p_1 + p_2)$. But, by our previous lemma, after cutting down by two projections, we have the direct sum of two semi-simple algebras is already maximal P_1, which contradicts that R will be maximal P_1.

□