

Algorithmic Algebraic Geometry
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Amoebas and their Tropical Varieties Writeup

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Abstract

Amoebas make obvious extraordinarily beautiful art simply within the polynomials of several variables. They allow for a unique perspective of the complex zero set, and have many remarkable properties. In particular, a recent result of Martin Avendaño expands our understanding of these constructions through tropical varieties, allowing us to observe the relationship between the Archimedean-Newton Polygons, tropical varieties, and the amoebas of polynomials. These provide an observation linking our logarithmic zero set with the geometric interpretation of a polynomial of several variables. For my project as part of the Texas A&M NSF sponsored Research Experience for Undergraduates in Mathematics, I implemented Avendaño's tropical variety to MATLAB, and explored its limitations to understand the precise significance of his new theorem.

1 Background on Amoebas and Avendaño's Theorem

Amoebas were first defined in a 1994 book by Gelfand, Kapranov, and Zelevinsky as the logarithm of the zero set of a polynomial of several variables, particularly interesting for their beautiful graphical representation. Amoebas also hold a significant relation to Tropical Geometry, especially to tropical varieties. In particular, the spine of an amoeba, which is created by shrinking the amoeba to its piecewise linear form is a tropical variety.

In a paper from 2011, Martin Avendaño lays out an exciting new theorem concerning amoebas and the tropical varieties defined by the inner normal fan of the ArchNewt lifting of f , that is for $f \in \mathbb{R}[x_1, \dots, x_n]$ $f(x) = \sum_{i=1}^t c_i x^{a_i}$, $ArchNewt(f) = \text{Lower Convex hull of } \{(a_1, -\log |c_1|), \dots, (a_t, -\log |c_t|)\}$. From this, we can consider the inner normal fan and get points of intersection with the hyperplane $\{x_{n+1} = 1\}$. In the case of $n = 2$, as I will be considering for my project, this is simply the plane $\{z = 1\}$. Avendaño then tells us that the Hausdorff distance, denoted by Δ , between the negative of the Amoeba and the tropical variety, $Trop(f)$, is $\Delta(-Amoeba(f), Trop(f)) \leq \log(t - 1)$. (Recall that t is the number of terms in the polynomial.)

This is a beautiful theorem for several reasons, first because of the way it demonstrates the usefulness of somewhat simple geometric intuition to better understand the general structure, as well as the way it links topics of slightly different areas through this nice metric. Finally, the simplicity is unavoidable; the theorem can be loosely explained by a brief sentence: for any point on the negative amoeba, there exists a point on this tropical variety within the logarithm of the number of terms of the polynomial minus one (and vice versa). Of particular use, however, is the other direction, since this tropical variety is much easier to compute than the amoeba.

2 Implementation of Avendaño's Theorem

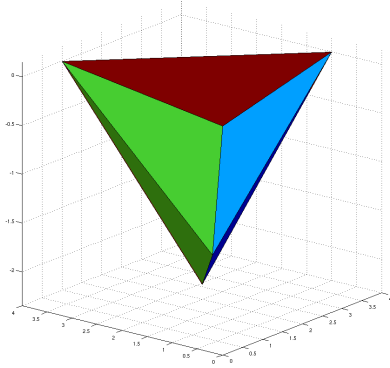
For my work in particular, I considered a slight modification to this theorem by merely switching his statement to read $\Delta(\text{Amoeba}(f), -\text{Trop}(f)) \leq \log(t - 1)$, simply because plotting the negative of an amoeba is much more difficult than plotting the negative of a tropical variety. To observe firsthand Avendaño's result in concrete examples, I considered a few examples by plotting an amoeba with a program and then by hand calculating the tropical variety he defined. This was painfully slow, unfortunately, and so I soon decided on a more effective approach of implementing his tropical variety as a function in MATLAB. With this implementation came the challenge of learning to use the program, as I came with basically no MATLAB or general programming knowledge. Time, however, built the knowledge, and soon I had a working program.

The program begins by calculating the lower convex hull defined by the ArchNewt lifting, and then computes the inner normal points defined by the intersection of the inner normal vectors of each face of this hull with the plane $z = 1$. Next, we must consider the faces sharing a common edge and connect the points corresponding to these faces with line segments before finding the exterior edges of the faces of the lower convex hull. Then the corresponding points were joined to lines perpendicular exterior of the of the negative Newton Polytope (Newt), that is, negative ArchNewt without the last coordinate. Finally, this was scaled to be representative of the size of the overall tropical variety.

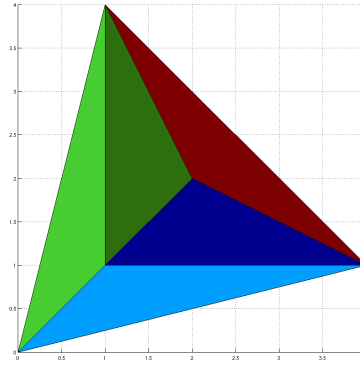
While clearly using the $\log(t-1)$ variation allowed, in many cases it seemed to nearly present the spine of the amoeba that is a representation of the Amoeba's fundamental shape. In the particular example shown in Figure 1, the correspondence to the amoeba itself is impressive, though not perfect. Looking at this particular example, it is clear that this tropical variety does not perfectly represent the spine of the amoeba, and thus has a slight error in completely representing the amoeba. From this observation, we are justified in asking: How accurate is this particular tropical variety in approximating the overall structure and shape of the amoeba?

Figure 1: $f(x) = x_1^4 x_2 + x_1 x_2^4 - 9x_1^2 x_2^2 + 5x_1 x_2 + 1$

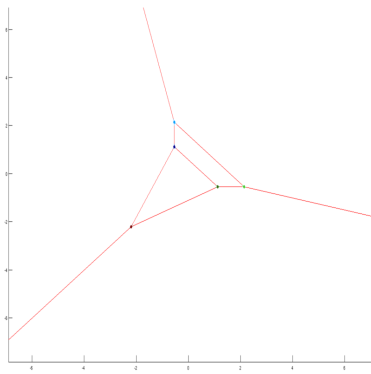
(a) $ArchNewt(f)$



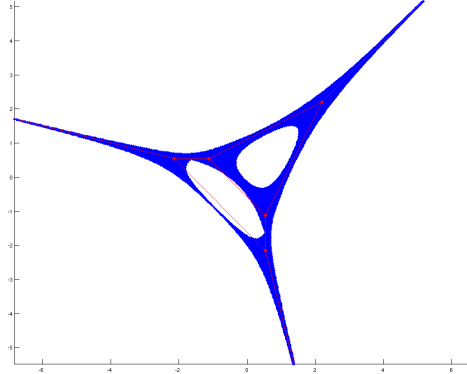
(b) $Newt(f)$ (triangulated to correspond with $ArchNewt$)



(c) $Trop(f)$



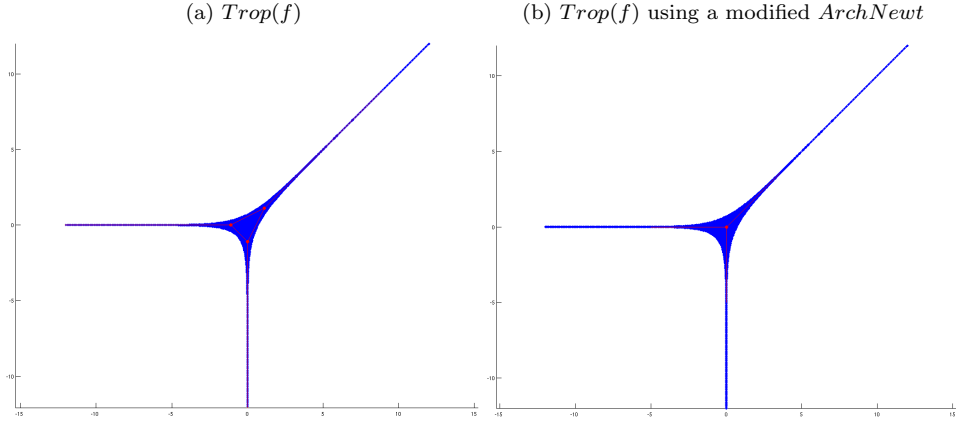
(d) $Amoeba(f)$ with $-Trop(f)$



3 Limitations in Representing Amoeba

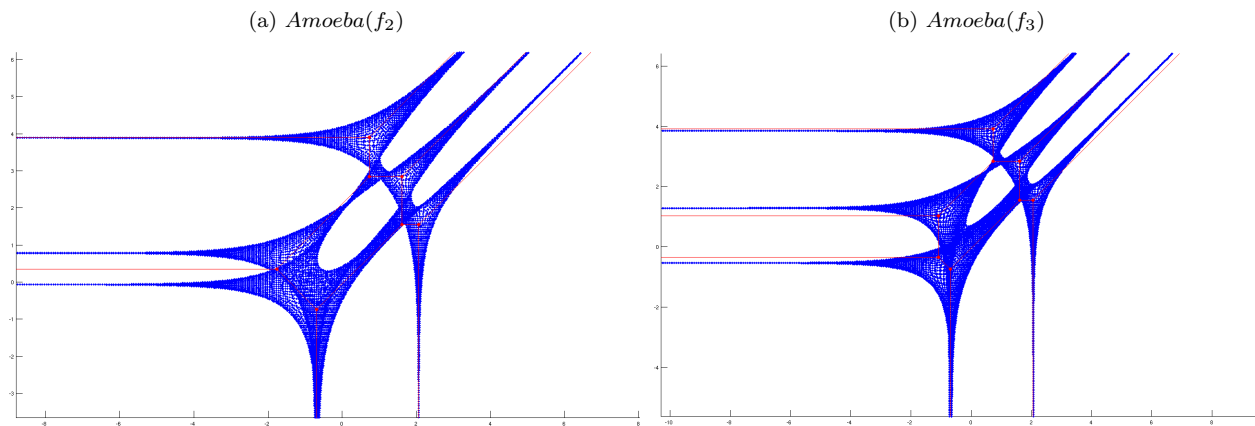
For my project specifically, I explored the limitations of Avendaño's tropical variety in representing the amoeba of a polynomial. The results were mixed. While in nearly every case, the amoeba logically looked similar to the tropical variety, there were things slightly incorrect in a few specific examples.

Figure 2: $f_1(x_1, x_2) = x_1^3 + x_2^3 - 3x_1x_2 + 1$



As we see in Figure 2, the amoeba has a no hole in its central mass, yet the $Trop(f)$ contains a hole. Clearly, this is a situation where the tropical variety Avendaño has defined does not accurately model the amoeba. Holes are important to the specific structure of the amoeba and so ought to appear in the tropical variety exactly when the amoeba has a hole. Under a modified choice of $ArchNewt$, however, the amoeba is perfectly represented by exactly the tropical variety of this. It is also clear that entire amoeba is within $\log(3)$ of the modified tropical variety, so Avendaño's maximum distance between the two holds.

Figure 3: $f_2(x_1, x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + 59x_2 - 100$
 $f_3(x_1, x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + 70)x_2 - 100$



In Figure 3(a), we see an example of an amoeba with three tentacles in

the negative x_1 direction, but only two tentacles on ArchNewt. This shows that though Avendaño's tropical variety is very close to the Amoeba (within $\log(9)$), it cannot completely accurately represent the tentacles of the Amoeba, and thus cannot guarantee the number of tentacles to be found on the Amoeba. By modifying the 9th term of f_2 to get f_3 , we can affect the lower convex hull created by the ArchNewt lifting to split a tentacle of the tropical variety and more accurately represent the tentacles of the amoeba. (Though this does change the amoeba simultaneously.)

These examples show that while Avendaño's theorem tells us that the amoeba will be very close to the tropical variety he defines, it is certainly not guaranteed to also be an accurate representation of the shape of the amoeba itself. It is not necessarily similar to the spine, and unfortunately does not reveal accurately the properties of the amoeba itself. It does, however, give us a close range of values to consider for the amoeba, and creates a rough approximation that is easy to compute and simple to compare.

4 Conclusion

The implications of these limitations are important to further study because they indicate that the amoebas cannot be completely represented in this way without some improvement to the definition of the tropical variety, likely through the alteration of the ArchNewt lifting. Yet the value of Avendaño's theorem is hard to state fully, because it confirms and demonstrates a simple connection between the Amoeba and this tropical variety that is defined through the ArchNewt. It means that while the tropical variety is not necessarily similar to the amoeba, it is within a distance of the amoeba, and therefore the amoeba is guaranteed to be contained within a certain distance of the tropical variety. This tells us much about where the amoeba lies within the plane, as it represents a significantly smaller portion of the plane than the entire plane. Though it is likely that tighter bounds may be found for amoebas and their particular varieties, the general simplicity and beauty of this theorem is astounding.

Overall, the limitations I observed suggest the existence of a modified lifting for ArchNewt that would adjust the convex hull to improve the lifting by making sections of the ArchNewt planar or nonplanar to adapt the inner normal to better represent the amoeba. Such a lifting, however, seems likely to be complicated, since in some cases, no modification is necessary to perfectly represent the amoeba, but in others, several points seem to need adjustment, or points would need to be added or removed. Going forward with the study of amoebas and tropical varieties, this will definitely remain an area of study, and a place to seek improvement to fully use the potential of these tropical varieties to represent the amoebas.

References

- [1] Israel M. Gelfand, Mikhail M. Kapranov, Andrei Zelevinsky: Discriminants, Resultants, and Multidimensional Determinants, 1994.
- [2] Martin Avendaño: On the Distance of a Complex Amoeba to its Tropical Variety, 2011.