# EFFECTIVE NON-VANISHING OF CLASS GROUP L-FUNCTIONS FOR BIQUADRATIC CM FIELDS

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Abstract. In this report, we outline a proof that, given co-prime, square-free integers $d_1 > 0$ and $d_2 < 0$ such that $|d_2| \geq (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1)+2\right\}$, there exists at least one class group character $\chi$ of the biquadratic CM field $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that the $L$-function $L(\chi, s)$ attached to this character is non-vanishing at $s = \frac{1}{2}$.

## 1. Introduction and Statement of Results

$L$-functions are important objects in number theory due to the deep arithmetic information that they encode. Some examples of $L$-functions are given by Dirichlet series,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $\{a_n\}$ is some arithmetic sequence of complex numbers and $s$ is a complex number with sufficiently large real part. For example, the prototypical $L$-function is the famous Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This sum can be expressed as an Euler product

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

revealing its connection to the prime numbers. In this paper, we will study an interesting family of $L$-functions called class group $L$-functions. In particular, we will prove an effective non-vanishing theorem for central values of these $L$-functions.

In order to describe these $L$-functions, we will need to introduce some notation and definitions. Let $d_1 > 0$ and $d_2 < 0$ be coprime, squarefree integers. Let $K = \mathbb{Q}(\sqrt{d_1})$ be a real quadratic field and let $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be an imaginary quadratic extension of $K$. Let $D_K, D_E$ be the absolute values of the discriminants of $K$ and $E$, respectively. Let $\mathcal{O}_K, \mathcal{O}_E$ be the rings of integers of $K, E$, respectively, $\mathcal{O}_K^\times$ be the group of units of $\mathcal{O}_K$, $Cl(\mathcal{O}_E)$ be the ideal class group of $E$, $h_E$ be the class number, and $\widehat{Cl(\mathcal{O}_E)}$ be the group of characters, and $R_K$ the regulator of $K$. Let $\zeta_K(s)$ denote the Dedekind zeta function, and $\gamma_K$ denote the constant term of the Laurent expansion of $\zeta_K(s)$ at $s = 1$. Note that $\gamma_{\mathbb{Q}}$ is the usual Euler's constant.

We will expand upon a few of these definitions. Recall that the *ideal class group* of a field $E$ is the quotient group $Cl(E) = \frac{J_E}{P_E}$ where $J_E$ is the set of fractional ideals of $E$ and $P_E$ is the set of principal fractional ideals. A fractional ideal $\mathfrak{a}$ is an $\mathcal{O}_E$ submodule of $E$ of the form $\mathfrak{a} = \frac{1}{x}\mathfrak{b}$, where $x$ is an algebraic integer and $\mathfrak{b} \subseteq \mathcal{O}_E$ is an integral ideal, and a principal ideal is an ideal generated by a single element. The ideal class group is a finite abelian group. It is useful because it measures how "close" $\mathcal{O}_E$ is to being a principal ideal domain. The class number, $h_E$, is the order of the class group. $\mathcal{O}_E$ is a principal ideal domain if and only if $E$ has class number 1.

**Definition 1.1.** Let $G$ be a finite abelian group. A function $\chi : G \to \mathbb{C}^\times$ is a *character* of $G$ if it is a group homomorphism. The set of characters $\widehat{G}$ is a finite abelian group called the *character group* of $G$.

We are now ready to define the class group $L$-function:

**Definition 1.2.** Given $\chi \in \widehat{Cl(\mathcal{O}_E)}$, we define the *class group $L$-function* by

$$L(\chi, s) = \sum_{[A] \in Cl(\mathcal{O}_E)} \chi(A) \zeta_E(s, A)$$

where

$$\zeta_E(s, A) = \sum_{0 \neq \mathfrak{a} \in [A]} N(\mathfrak{a})^{-s}$$

is the *partial zeta function* (here, $N(\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$).

It is well known that if the character $\chi$ is nontrivial, $L(\chi, s)$ extends to an entire function on the complex plane $\mathbb{C}$, which satisfies the following functional equation:

$$\Lambda_K(\chi, s) = \Lambda_K(\chi, 1 - s)$$

where $\Lambda_K(\chi, s) := \left(\frac{\sqrt{D_K}}{(2\pi)^2}\right)^s \Gamma(s)^2 L(\chi, 2)$ (see, for example, [Mas, Sec. 1]). This functional equation tells us that $L(\chi, s)$ is symmetric about the point $s = \frac{1}{2}$. Thus, $L(\chi, \frac{1}{2})$ is called the central value.

We now define the Hilbert modular Eisenstein series associated to $K$. Since $K$ is a real quadratic field, it has two embeddings, $\sigma_1$ and $\sigma_2$, into the real numbers.
Let

$$\mathbb{H} = \{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}$$

be the complex upper half-plane and $z = (z_1, z_2) \in \mathbb{H}^2$ where $z_j = x_j + iy_j \in \mathbb{H}$. Let $y = \mathrm{Im}(z) = (y_1, y_2)$. Then

$$N(y) = \prod_{j=1}^{2} y_j$$

and

$$N(\alpha + \beta z) = \prod_{j=1}^{2} (\sigma_j(\alpha) + \sigma_j(\beta) z_j)$$

for $\alpha, \beta \in K$.

**Definition 1.3.** The *Hilbert modular Eisenstein series* is defined by

$$E_K(z, s) = \sum_{0 \neq (\alpha, \beta) \in \mathcal{O}_K^2 / \mathcal{O}_K^\times} \frac{N(y)^s}{|N(\alpha + \beta z)|^{2s}}, \quad z \in \mathbb{H}^2, \quad \mathrm{Re}(s) > 1.$$

We now state a useful result that connects this Eisenstein series to the average of class group $L$-functions.

**Proposition 1.4.** *For $\chi \in \widehat{Cl(\mathcal{O}_E)}$, we have*

$$\frac{1}{h_E}\sum_{\chi\in\widehat{Cl(\mathcal{O}_E)}} L(\chi,s) = \left(\frac{4D_K}{\sqrt{D_E}}\right)^s \frac{1}{[\mathcal{O}_E^\times : \mathcal{O}_K^\times]} E_K(z_{\mathcal{O}_E}, s),$$

*where $z_{\mathcal{O}_E} \in \mathbb{H}^2$ is the CM point associated to the class $[\mathcal{O}_E]$ (to be defined later).*

Since $\left(\frac{4D_K}{\sqrt{D_E}}\right)^s \frac{1}{[\mathcal{O}_E^\times:\mathcal{O}_K^\times]}$ is always non-zero, in order to show that there exists at least one $\chi$ such that $L(\chi,\frac{1}{2}) \neq 0$, it suffices to show that $E_K(z_{\mathcal{O}_E},\frac{1}{2}) \neq 0$.

**Main Theorem** (Theorem 1)**.** *Let $d_1 > 0$ and $d_2 < 0$ be square-free, co-prime integers with $d_1 \equiv 1 \mod 4$ and $d_2 \equiv 2$ or $3 \mod 4$. Assume $K = \mathbb{Q}(\sqrt{d_1})$ has narrow class number 1 and let $E = \mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$. Then if*

$$|d_2| \geq (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1)+2\right\},$$

*there exists a character $\chi \in \widehat{Cl(\mathcal{O}_E)}$ such that $L(\chi,\frac{1}{2}) \neq 0$.*

Essentially, this theorem reduces the question of whether these class group $L$-functions are non-vanishing to a finite (albeit large) calculation. We will outline our proof in the next section.

## 2. Proof Outline

A crucial step is the following decomposition of the Eisenstein series.

**Proposition 1.** *We have*

$$E_K(z,\tfrac{1}{2}) = M(z,\tfrac{1}{2}) + H(z,\tfrac{1}{2})$$

*where*

$$M(z,\tfrac{1}{2}) = \sqrt{N(y)}\left[c_{-1}\log(N(y)) - c_{-1}\log\left(\frac{\pi^2}{D_K}\right) + 2\gamma_K - 2c_{-1}(\gamma_{\mathbb{Q}} + \log(4))\right]$$

*and*

$$H(z,\tfrac{1}{2}) = \sqrt{N(y)}\sum_{\gamma\in\mathcal{O}_K}\sum_{0\neq\nu\in\mathcal{O}_K^\vee} c_\nu(\gamma y)e^{2\pi i Tr(\gamma\nu x)}.$$

*Here, $c_{-1} = \frac{2R_K}{\sqrt{d_1}}$ is the residue of the Dedekind zeta function at $s=1$.*

Noting that $D_K = d_1$ and $D_E = |d_2|$ under our assumptions, the average formula becomes

$$\frac{1}{h_E}\sum_{\chi\in\widehat{Cl(\mathcal{O}_E)}} L(\chi,\tfrac{1}{2}) = \left(\frac{4d_1}{\sqrt{|d_2|}}\right)^{\frac{1}{2}} \frac{1}{[\mathcal{O}_E^\times : \mathcal{O}_K^\times]} E_K(z_{\mathcal{O}_E},\tfrac{1}{2})$$

where the special point at which the Eisenstein Series is evaluated is

$$z_{\mathcal{O}_E} = \left(\sqrt{d_2},\sqrt{d_2}\right) \in \mathbb{H}^2.$$

Then $\mathrm{Im}(z_{\mathcal{O}_E}) = y = \left(\sqrt{|d_2|}, \sqrt{|d_2|}\right)$ and $N(y) = |d_2|$. Also, under these conditions, using Proposition 1, we can write

$$E_K(z_{\mathcal{O}_E}, \tfrac{1}{2}) = M(d_1, d_2) + H(d_1, d_2)$$

where

$$M(d_1, d_2) = \sqrt{|d_2|}\left[\frac{2R_K}{\sqrt{d_1}}\left(\log\left(|d_2|\right) - \log\left(\frac{\pi^2}{d_1}\right) - 2(\gamma_{\mathbb{Q}} + \log(4))\right) + 2\gamma_K\right]$$

and

$$H(d_1, d_2) = \sqrt{|d_2|}\sum_{\gamma\in\mathcal{O}_K}\sum_{0\neq\nu\in\mathcal{O}_K^\vee} c_\nu(\gamma y(z_{\mathcal{O}_E}))e^{2\pi i Tr(\gamma\nu x)}.$$

The vast majority of the project consisted of proving the following proposition, which is a difficult technical refinement of the proof of [Bau, Lemma 1].

**Proposition 2.1.** *If*

$$|d_2| \geq (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1) + 2)\right\},$$

*then*

$$|H(d_1, d_2)| \leq 6.80 \times 10^{-401}.$$

By a straightforward argument using the bound

$$\gamma_K > -2\frac{R_K}{\sqrt{d_1}}(\log(\sqrt{d_1}) - \gamma_{\mathbb{Q}} - \log(4\pi) + 1)$$

(which follows as a consequence of [Iha, Sec. 1.1]), and the well-known bound

$$R_K > \log(2\sqrt{d_1}),$$

we show that $M(d_1, d_2) > 1$, assuming our lower bound on $|d_2|$.

Thus, we have $|M(d_1, d_2)| > |H(d_1, d_2)|$, hence $|E_K(z_{\mathcal{O}_E}, \frac{1}{2})| > 0$. This completes the outline of the proof.

**Example 2.2.** If $d_1 = 5$, then for $|d_2| \geq 2.77028 \times 10^{13}$, there exists a $\chi \in \widehat{Cl(\mathcal{O}_E)}$ such that $L(\chi, \frac{1}{2}) \neq 0$.

**Remark 2.3.** Theorem 1 can be generalized to *any* CM extension $E$ of a totally real field $K$ with narrow ideal class number one. The details will appear in a future paper.

## References

[Bau] Hartmut Bauer, *Zeros of Asai-Eisenstein series*. Math. Z. **254** (2006), no. 2, 219-237.

[Iha] Yasutaka Ihara, *The Euler-Kronecker invariants in various families of global fields*. Séminaires et Congrés. **21** (2009), 79-102.

[Mas] R. Masri, *CM Cylces and Nonvanishing of Class Group L-Functions*. Math. Res. Lett. **17** (2012), no. 04, 749-760.