Generalized Dedekind Sums Arising from Eisenstein Series

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Mobius Transformations

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Given $\gamma \in SL_2(\mathbb{Z})$, the *Mobius transformation* associated to γ is the complex map defined by

$$z \to \frac{az + b}{cz + d},$$

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We write

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   (complex analytic, harmonic functions, ...).
3. f exhibits some boundary behavior.
   (polynomial growth, boundedness as function approaches $i\infty$, ...)

Eisenstein Series

For $k \geq 4$ and k even, the weight-k *Eisenstein Series* is

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For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$E_k(\gamma z) = (cz+d)^k E_k(z).$$

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Example: Jacobi/Legendre Symbols

Eisenstein Series with Dirichlet Characters

$$E_{\chi_1,\chi_2}(z,s) = \frac{1}{2} \sum_{\gcd(c,d)=1} \frac{(q_2 y)^s \chi_1(c)\chi_2(d)}{|cq_2 z + d|^{2s}} \Big(\frac{|cq_2 z + d|}{cq_2 z + d} \Big)^k$$

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$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}$$

Periodicity of E_{χ_1,χ_2}

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(q_1 q_2)$.

Then

$$Tz = \frac{1z+1}{0z+1} = z+1,$$

so

$$\begin{aligned} E_{\chi_1,\chi_2}(z+1,s) &= (0z+1)^k \chi_1(1)\bar{\chi_2}(1) E_{\chi_1,\chi_2}(z,s) \\ &= E_{\chi_1,\chi_2}(z,s). \end{aligned}$$

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Thus, E_{χ_1,χ_2} is periodic.

Fourier Expansion for the Completed Eisenstein Series

Define the *completed Eisenstein series* as

$$E^*_{\chi_1,\chi_2}(z,s) := \frac{(q_2/\pi)^s}{i^{-k}\tau(\chi_2)}\Gamma(s+\tfrac{k}{2})L(2s,\chi_1\chi_2)E_{\chi_1,\chi_2}(z,s)$$

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The Fourier expansion for the completed Eisenstein series is

$$\begin{aligned} E^*_{\chi_1,\chi_2}(z,s) &= e^*_{\chi_1,\chi_2}(y,s) + \sum_{n\neq 0} \frac{\lambda_{\chi_1,\chi_2}(n,s)}{\sqrt{|n|}} e^{2\pi i n x} \\ &\quad \cdot \frac{\Gamma(s+\frac{k}{2})}{\Gamma(s+\frac{k}{2}sgn(n))} W_{\frac{k}{2}sgn(n),s-\frac{1}{2}}(4\pi|n|y). \end{aligned}$$

Evaluating $E^*_{\chi_1,\chi_2}(z,s)$ at $k=0$ and $s=1$

$$
\begin{aligned}
E^*_{\chi_1,\chi_2}(z,1) \quad &= \quad \sum_{n>0} \frac{e^{2\pi i n z}}{\sqrt{n}} \sum_{ab=n} \chi_1(a)\overline{\chi_2(b)} \left(\frac{b}{a}\right)^{\frac{1}{2}} \\
&\quad + \chi_2(-1) \sum_{n>0} \frac{e^{-2\pi i n \bar{z}}}{\sqrt{n}} \sum_{ab=n} \chi_1(a)\overline{\chi_2(b)} \left(\frac{b}{a}\right)^{\frac{1}{2}}
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The η-function and Dedekind Sums

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$$

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$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\Big(\frac{hr}{k} - \Big[\frac{hr}{k}\Big] - \frac{1}{2}\Big)$$

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We have been investigating the function f_{χ_1,χ_2}.

Transformation Properties of $f_{\chi_1,\chi_2}(z)$

Define

$$\phi_{\chi_1,\chi_2}(\gamma, z) := f_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) f_{\chi_1,\chi_2}(z).$$

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Main Goal. Find a finite sum formula for ϕ_{χ_1,χ_2}.

Properties of ϕ_{χ_1,χ_2}

Lemma 1. *The function* ϕ_{χ_1,χ_2} *is independent of* z.

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Proof. Since $E^*_{\chi_1,\chi_2}(\gamma z, 1) = \psi(\gamma) E^*_{\chi_1,\chi_2}(z, 1)$ and $E^*_{\chi_1,\chi_2}(z, 1) = f_{\chi_1,\chi_2}(z) + \chi_2(-1)\overline{f_{\overline{\chi_1},\overline{\chi_2}}}(z)$,

$$\phi_{\chi_1,\chi_2}(\gamma, z) = -\chi_2(-1)\overline{\phi_{\overline{\chi_1},\overline{\chi_2}}}(\gamma, z).$$

Since ϕ_{χ_1,χ_2} is a holomorphic function and $\overline{\phi_{\overline{\chi_1},\overline{\chi_2}}}$ is an antiholomorphic function, ϕ_{χ_1,χ_2} must be constant. □

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- From now on, we will write $\phi_{\chi_1,\chi_2}(\gamma)$ instead of $\phi_{\chi_1,\chi_2}(\gamma, z)$.

Properties of ϕ_{χ_1,χ_2}

Lemma 2. *Let* $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$. *Then*

$$\phi_{\chi_1,\chi_2}(\gamma_1\gamma_2) = \phi_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)\phi_{\chi_1,\chi_2}(\gamma_2).$$

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$$\phi_{\chi_1,\chi_2}(\gamma_1\gamma_2) = \phi_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)\phi_{\chi_1,\chi_2}(\gamma_2).$$

Proof. Since ψ is multiplicative,

$$\begin{aligned}
\phi_{\chi_1,\chi_2}(\gamma_1\gamma_2) &= f_{\chi_1,\chi_2}(\gamma_1\gamma_2 z) - \psi(\gamma_1\gamma_2) f_{\chi_1,\chi_2}(z) \\
&= f_{\chi_1,\chi_2}(\gamma_1\gamma_2 z) - \psi(\gamma_1)\psi(\gamma_2) f_{\chi_1,\chi_2}(z) \\
&= f_{\chi_1,\chi_2}(\gamma_1\gamma_2 z) - \psi(\gamma_1) f_{\chi_1,\chi_2}(\gamma_2 z) \\
&\quad + \psi(\gamma_1) f_{\chi_1,\chi_2}(\gamma_2 z) - \psi(\gamma_1)\psi(\gamma_2) f_{\chi_1,\chi_2}(z) \\
&= \phi_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)\phi_{\chi_1,\chi_2}(\gamma_2). \qquad \square
\end{aligned}$$

Main Theorem

Theorem. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$. *Then*

$$\phi_{\chi_1,\chi_2}(\gamma) = \frac{-\pi i \chi_2(-1)}{\tau(\overline{\chi_1})} \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} - \frac{aj}{c}\right),$$

where

$$B_1(z) = \begin{cases} z - \lfloor z \rfloor - \frac{1}{2}, & z \notin \mathbb{Z} \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\tau(\chi) = \sum_{n=0}^{q-1} \chi(n) e^{\frac{2\pi i n}{q}},$$

for χ *modulo* q.

Carnival Funhouse Proof of Main Theorem

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1q_2)$. Choose $z = \frac{-d}{c} + \frac{i}{c^2u} \in \mathbb{H}$ for some $u \in \mathbb{R}$, $u \neq 0$. Then $\gamma z = \frac{a}{c} + iu$.

$$\phi_{\chi_1,\chi_2}(\gamma) = \lim_{u \to 0^+} \left(f_{\chi_1,\chi_2}\left(\frac{a}{c} + iu\right) - \psi(\gamma) f_{\chi_1,\chi_2}\left(\frac{-d}{c} + \frac{i}{c^2u}\right)\right)$$

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$$\lim_{u \to 0^+} f_{\chi_1,\chi_2}\left(\frac{-d}{c} + \frac{i}{c^2 u}\right) = 0.$$

Thus,

$$\phi_{\chi_1,\chi_2}(\gamma) = \lim_{u \to 0^+} f_{\chi_1,\chi_2}\left(\frac{a}{c} + iu\right).$$

Carnival Funhouse Proof of Main Theorem

$$f_{\chi_1,\chi_2}(z) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\chi_1(l)\overline{\chi_2}(k)}{l} e^{2\pi i k l z}.$$

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$$f_{\chi_1,\chi_2}(z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\chi_1(l)\overline{\chi_2}(k)}{l} e^{2\pi i k l z}.$$

Simplifying f_{χ_1,χ_2} and evaluating $\lim_{u\to 0^+} f_{\chi_1,\chi_2}\left(\frac{a}{c}+iu\right)$, we get

$$\phi_{\chi_1,\chi_2}(\gamma) = \chi_2(-1)\sum_{l=1}^{\infty}\frac{\chi_1(l)}{l}\sum_{j \pmod{c}} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) e^{\frac{-2\pi i a l j}{c}}.$$

Carnival Funhouse Proof of Main Theorem

From the transformation properties of $E^*_{\chi_1,\chi_2}$, we have

$$\phi_{\chi_1,\chi_2}(\gamma) = \frac{1}{2}(\phi_{\chi_1,\chi_2}(\gamma) - \chi_2(-1)\overline{\phi_{\overline{\chi_1},\overline{\chi_2}}(\gamma)}).$$

We simplify this more symmetric version of ϕ_{χ_1,χ_2} to get

$$\phi_{\chi_1,\chi_2}(\gamma) = \frac{-\pi i \chi_2(-1)}{\tau(\overline{\chi_1})} \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} - \frac{aj}{c}\right).$$

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$$12hks(h,k) + 12khs(k,h) = h^2 + k^2 - 3hk + 1$$

References

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