BOUNDS FOR COEFFICIENTS OF THE $f(q)$ MOCK THETA FUNCTION AND APPLICATIONS TO PARTITION RANKS

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Abstract. We compute effective bounds for $\alpha(n)$, the Fourier coefficients of Ramunujan's mock theta function $f(q)$ utilizing a finite algebraic formula due to Brunier and Schwagenscheidt [1]. We then use these bounds to prove a conjecture of Hou and Jagadeesan [2] on the convexity of the even and odd partition rank counting functions.

1. Introduction and Statement of Results

For a nonnegative integer n, a partition of n is a finite list of nondecreasing positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The partition number $p(n)$ denotes the number of partitions of n which has been of large interest to number theorists.

Given a partition λ of n, we can define the rank of λ as $\lambda_k - k$. In words, this is the largest part of the partition minus the number of parts. For any n, we can consider $N(r, t; n)$ which counts the number of partitions of n that have rank equal to $r \pmod{t}$.

For the case of $t = 2$, we analyze partitions with even or odd rank, captured by the coefficients $\alpha(n)$ of Ramanujan's mock theta function

$$\begin{aligned} f(q) &:= 1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \ldots (1+q^n)^2} \\ &= 1 + \sum_{n=0}^{\infty} \alpha(n) q^n \end{aligned}$$

for $q := e^{2\pi i z}$, where $\alpha(n) = N(0, 2; n) - N(1, 2; n)$.

In this paper, we will prove the following asymptotic formula for $\alpha(n)$ with an effective bound on the error term:

Theorem 1.1. *Let $D_n := -24n+1$ be the fundamental discriminant and $l(n) := \pi\sqrt{|D_n|}/6$. Then for all $n \geq 1$,*

$$\alpha(n) = (-1)^{n+1} \frac{\sqrt{6}}{\sqrt{24n-1}} e^{l(n)/2} + E(n)$$

where

$$|E(n)| < (4.30 \times 10^{23}) 2^{q(n)} |D_n|^2 e^{l(n)/3}$$

with

$$q(n) := \frac{\log(|D_n|)}{|\log\log(|D_n|) - 1.1714|}$$

In 1966, Andrews and Dragonette [3, pp. 456] conjectured a Rademacher-type infinite series for $\alpha(n)$. This conjecture was proved by Bringmann and Ono [4], who obtained the following formula:

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{1/2}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) \tag{1.1}$$

where $A_{2k}(n)$ is a certain twisted Kloosterman-type sum and $I_{1/2}$ is the I-Bessel function of order 1/2. One can easily show that the $k = 1$ term in (1.1) agrees with the main term in Theorem 1.1. Since (1.1) is only conditionally convergent, it is very difficult to bound. Using a different, finite algebraic formula for $\alpha(n)$ due to Alfes [5], Masri [6, Theorem 1.3] gave an asymptotic formula for $\alpha(n)$ with a power-saving error term. The exponent in this bound was later improved by Ahlgren and Dunn [7, Theorem 1.3] by bounding the series (1.1) directly.

Using Theorem 1.1, we look to show a certain convexity property for $N(r, 2; n)$. In particular, we aim to prove a conjecture of Hou and Jagadeesan in [2]

Conjecture 1. *If $r = 0$ (resp. $r = 1$), then we have that*

$$N(r, 2; a)N(r, 2; b) > N(r, 2; a + b)$$

for all $a, b \geq 11$ (resp 12).

Hou and Jagadeesan [2, Theorem 1.1] proved a similar convexity bound modulo 3; however, their techniques do not extend to modulus two. Here, we overcome these difficulties using Theorem 1.1 and prove the following:

Theorem 1.2. *Conjecture 1 is true.*

We also demonstrate effective equidistribution of partition ranks modulo 2, improving upon the results of Masri [6] and Males [8] (see Corollary 5.2). Masri proved equidistribution of partition ranks modulo 2 with a power-saving error term, however his results were not effective, and so could not be applied towards Conjecture 1.

We now describe our approach to Theorem 1.1. To give an effective bound on the error term for $\alpha(n)$, we will use a formula for $\alpha(n)$ which expresses it as a trace over singular moduli. To state this formula, consider the weight zero weakly-holomorphic modular form for $\Gamma_0(6)$ defined by

$$F(z) := -\frac{1}{40}\frac{E_4(z) + 4E_4(2z) - 9E_4(3z) - 36E_4(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2} = q^{-1} - 4 - 83q - 296q^2 + \ldots. \tag{1.2}$$

Brunier and Schwagenscheidt [1, Theorem 3.1] proved

Theorem (Brunier/Schwagenscheidt)**.** *For $n \geq 1$, we have*

$$\alpha(n) = -\frac{1}{\sqrt{|D_n|}}\mathrm{Im}(S(n))$$

where

$$S(n) := \sum_{[Q]} F(\tau_Q).$$

Here, the sum is over the $\Gamma_0(6)$ equivalence classes of discriminant D_n positive definite, integral binary quadratic forms $Q = [a, b, c]$ such that $6 \mid a$ and $b \equiv 1 \pmod{12}$, and τ_Q is the Heegner point given by the root $Q(\tau_Q, 1)$ in the complex upper half-plane $\mathbb{H}$.

Our proof of Theorem 1.1 is inspired by work of Locus-Dawsey and Masri [9], who used a similar algebraic formula due Ahlgren and Andersen [10] for the Andrews smallest-parts function to give an asymptotic formula for $\mathrm{spt}(n)$ with an effective bound on the error term and prove several conjectural inequalities of Chen [11].

Organization. The paper is organized as follows. In Section 2, we review some facts regarding quadratic forms and Heegner points. In Section 3, we derive the Fourier expansion of $F(z)$ and effective bounds on its coefficients. In Section 4, we prove Theorem 1.1. In Section 5, we discuss corollaries to Theorem 1.1. Finally, in Section 6, we prove Theorem 1.2.

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2. Quadratic Forms and Heegner Points

Let N be a positive integer and D be a negative integer discriminant coprime to N. Let $\mathcal{Q}_{D,N}$ be the set of positive definite, integral binary quadratic forms

$$Q(X,Y) = [a,b,c](X,Y) = aX^2 + bXY + cY^2$$

with discriminant $b^2 - 4ac = D < 0$ with $a \equiv 0 \pmod{N}$. The congruence subgroup $\Gamma_0(N)$ acts on $\mathcal{Q}_{D,N}$ by

$$Q \circ \sigma = [a^\sigma, b^\sigma, c^\sigma]$$

with $\sigma = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in \Gamma_0(N)$, where

$$\begin{aligned} a^\sigma &= aw^2 + bwy + cy^2 \\ b^\sigma &= 2awx + b(wz + xy) + 2cyz \\ c^\sigma &= ax^2 + bxz + cz^2. \end{aligned}$$

Given a solution $r \pmod{2N}$ of $r^2 \equiv D \pmod{4N}$, we define the subset of forms

$$\mathcal{Q}_{D,N,r} := \{Q = [a,b,c] \in \mathcal{Q}_{D,N} \mid b \equiv r \pmod{2N}\}.$$

We can also consider the subset $\mathcal{Q}_{D,N}^{\mathrm{prim}}$ of primitive quadratic forms in $\mathcal{Q}_{D,N}$. These are the forms such that

$$\gcd(a,b,c) = 1.$$

We see that $\Gamma_0(N)$ also acts on $\mathcal{Q}_{D,N}^{\mathrm{prim}}$ and the number of $\Gamma_0(N)$ equivalence classes is given by the class number $h(D)$.

To each form $Q \in \mathcal{Q}_{D,N}$, we associate a Heegner point τ_Q which is the root of $Q(X,1)$ given by

$$\tau_Q = \frac{-b + \sqrt{D}}{2a} \in \mathbb{H}.$$

The Heegner points τ_Q are compatible with the action of $\Gamma_0(N)$ in the sense that if $\sigma \in \Gamma_0(N)$, then

$$\sigma(\tau_Q) = \tau_{Q \circ \sigma^{-1}}. \tag{2.1}$$

3. Fourier Expansion of $F(z)$

Let $D_n = -24n+1$ for $n \in \mathbb{Z}^+$ and define the trace of $F(z)$ by

$$S(n) := \sum_{[Q]\in \mathcal{Q}_{D_n,6,1}/\Gamma_0(6)} F(\tau_Q).$$

Proceeding as in [9, Section 3], we decompose $S(n)$ as a linear combination involving traces of primitive forms. Let $\Delta < 0$ be a discriminant with $\Delta \equiv 1 \pmod{24}$ and define the class polynomials

$$H_\Delta(X) := \prod_{[Q]\in \mathcal{Q}_{\Delta,6,1}/\Gamma_0(6)} (X - F(\tau_Q))$$

and

$$\widehat{H}_{\Delta,r}(F;X) := \prod_{[Q]\in \mathcal{Q}^{\text{prim}}_{\Delta,6,r}/\Gamma_0(6)} (X - F(\tau_Q)).$$

Let $\{W_\ell\}_{\ell|6}$ be the group of Atkin-Lehner operators for $\Gamma_0(6)$. We have by [1, pp. 47]

$$F|_0 W_\ell = \beta(\ell) F \tag{3.1}$$

where $\beta(\ell) = 1$ if $\ell = 1,2$ and $\beta(\ell) = -1$ if $\ell = 3,6$.

Using these eigenvalues we modify [12, Lemma 3.7] to get the following:

Lemma 3.1. *We have the decomposition*

$$H_\Delta(X) = \prod_{\substack{u>0\\ u^2|\Delta}} \varepsilon(u)^{h(\Delta/u^2)} \widehat{H}_{\Delta/u^2,1}(F; \varepsilon(u)X)$$

where $\varepsilon(u) = 1$ *if* $u \equiv 1,7 \pmod{12}$ *and* $\varepsilon(u) = -1$ *if* $u \equiv 5,11 \pmod{12}$.

Comparing coefficients on both sides of Lemma 3.1 yields the decomposition

$$S(n) = \sum_{\substack{u>0\\ u^2|D_n}} \varepsilon(u) S_u(n) \tag{3.2}$$

where

$$S_u(n) := \sum_{[Q]\in \mathcal{Q}^{\text{prim}}_{D_n/u^2,6,1}/\Gamma_0(6)} F(\tau_Q).$$

We now express $S_u(n)$ as a trace involving primitive forms of level 1. As in [9, Section 3], we let $\mathbf{C}_6$ denote the following set of right coset representatives of $\Gamma_0(6)$ in $SL_2(\mathbb{Z})$:

$$\gamma_\infty := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\gamma_{1/3,r} := \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad r = 0,1$$

$$\gamma_{1/2,s} := \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad s = 0,1,2$$

$$\gamma_{0,t} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t = 0,1,2,3,4,5$$

where $\gamma_\infty(\infty)$, $\gamma_{1/3,r}(\infty) = 1/3$, $\gamma_{1/2,s}(\infty) = 1/2$, and $\gamma_{0,t}(\infty) = 0$.

Recall that a form $Q = [a_Q, b_Q, c_Q] \in \mathcal{Q}_{\Delta,1}$ is reduced if

$$|b_Q| \le a_Q \le c_Q,$$

and if either $|b_Q| = a_Q$ or $a_Q = c_Q$, then $b_Q \ge 0$. Let $\mathcal{Q}_\Delta$ denote a set of primitive, reduced forms representing the equivalence classes in $\mathcal{Q}^{\text{prim}}_{\Delta,1}/SL_2(\mathbb{Z})$. For each $Q \in \mathcal{Q}_\Delta$, there is a unique choice of representative $\gamma_Q \in \mathbf{C}_6$ such that

$$[Q \circ \gamma_Q^{-1}] \in \mathcal{Q}^{\text{prim}}_{\Delta,6,1}/\Gamma_0(6).$$

This induces a bijection

$$\begin{aligned} \mathcal{Q}_\Delta &\longrightarrow \mathcal{Q}^{\text{prim}}_{\Delta,6,1}/\Gamma_0(6) \\ Q &\longmapsto [Q \circ \gamma_Q^{-1}]; \end{aligned} \tag{3.3}$$

see [13, pp. 505], or more concretely, [14, Lemma 3], where an explicit list of the matrices $\gamma_Q \in \mathbf{C}_6$ is given.

Using the bijection (3.3) and the compatibility relation (2.1) for Heegner points, the trace $S_u(n)$ can be expressed as

$$S_u(n) = \sum_{[Q] \in \mathcal{Q}^{\text{prim}}_{D_n/u^2,6,1}/\Gamma_0(6)} F(\tau_Q) = \sum_{Q \in \mathcal{Q}_{D_n/u^2}} F(\gamma_Q(\tau_Q)). \tag{3.4}$$

Therefore, to study the asymptotic distribution of $S_u(n)$, we need the Fourier expansion of $F(z)$ with respect to $\gamma_\infty, \gamma_{1/3,r}, \gamma_{1/2,s}$, and $\gamma_{0,t}$.

We first find the Fourier expansion of $F(z)$ at the cusp ∞.

Proposition 3.2. *The Fourier expansion of $F(z)$ at the cusp ∞ is*

$$F(z) = \sum_{n=-1}^{\infty} a(n) e(nz)$$

where $a(-1) = 1$, $a(0) = -4$ and for $n \ge 1$,

$$a(n) = \frac{2\pi}{\sqrt{n}} \sum_{\ell | 6} \frac{\beta(\ell)}{\sqrt{\ell}} \sum_{\substack{c>0 \\ c \equiv 0 \ (mod\ 6/\ell) \\ (c,\ell)=1}} c^{-1} S(-\bar{\ell}, n; c) I_1\left(\frac{4\pi\sqrt{n}}{c\sqrt{\ell}}\right),$$

where

$$\beta(\ell) := \begin{cases} 1, & \ell = 1,2 \\ -1, & \ell = 3,6, \end{cases}$$

I_1 is the I-Bessel function of order 1, and $S(a,b;c)$ is the ordinary Kloosterman sum defined as follows

$$S(a,b;c) := \sum_{\substack{d \pmod c \\ (c,d)=1}} e\left(\frac{a\bar{d} + bd}{c}\right),$$

$\bar{d}$ is the multiplicative inverse of d $(\text{mod } c)$.

Proof. Define the function

$$\mathcal{P}_F(z) := 2\sum_{\ell|6} \beta(\ell) F_1(W_\ell z, 1, 0)$$

where $F_1(z,1,0)$ is the Poincare series

$$F_1(z,1,0) := \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} [M_{0,1/2}(4\pi y)e(-x)]\,|_0\,\gamma$$

for $M_{\kappa,\mu}$ the usual Whittaker function. Then by a straightforward calculation, we have

$$\mathcal{P}_F(z) := 2\sum_{\ell|6}\beta(\ell) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} g(\gamma W_\ell z)$$

where

$$g(z) := \psi(y)e(-z),$$

and

$$\psi(y) := \pi\sqrt{y} I_{1/2}(2\pi y)e^{-2\pi y}.$$

Then arguing as in [15, Section 2], we get the Fourier expansion

$$\mathcal{P}_F(z) = e(-z) - e(-\bar{z}) + b_F(0) + \sum_{n=1}^{\infty} b_F(-n)e(-n\bar{z}) + \sum_{n=1}^{\infty} b_F(n)e(nz), \tag{3.5}$$

where

$$b_F(0) := 4\pi^2 \sum_{\ell|6} \frac{\beta(\ell)}{\ell} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/\ell} \\ (c,\ell)=1}} c^{-2} S(-\bar{\ell}, 0; c),$$

and for $n>0$

$$b_F(-n) := \frac{2\pi}{\sqrt{n}} \sum_{\ell|6} \frac{\beta(\ell)}{\sqrt{\ell}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/\ell} \\ (c,\ell)=1}} c^{-1} S(-\bar{\ell}, -n; c) J_1\left(\frac{4\pi\sqrt{n}}{c\sqrt{\ell}}\right),$$

and

$$b_F(n) := \frac{2\pi}{\sqrt{n}} \sum_{\ell|6} \frac{\beta(\ell)}{\sqrt{\ell}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/\ell} \\ (c,\ell)=1}} c^{-1} S(-\bar{\ell}, n; c) I_1\left(\frac{4\pi\sqrt{n}}{c\sqrt{\ell}}\right).$$

By (1.2), we have $a(-1)=1$ and $a(0)=-4$ so that

$$F|_0\gamma_\infty(z) = e(-z) - 4 + \sum_{n=1}^{\infty} a(n)e(nz).$$

The Atkin-Lehner operators for $\Gamma_0(6)$ are given by

$$W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}, \quad W_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}, \quad W_6 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}.$$

For each $\ell \mid 6$ and $v = 6/\ell$, let $V_\ell = \sqrt{\ell}W_\ell$ and

$$A_\ell = \begin{pmatrix} \frac{1}{\text{width of the cusp } 1/v} & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

cusp $1/v$	$\infty \simeq 1/6$	$1/3$	$1/2$	$0 \simeq 1$
ℓ	1	2	3	6
V_ℓ	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}$
A_ℓ	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/6 & 0 \\ 0 & 1 \end{pmatrix}$
$V_\ell A_\ell$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Proceeding as in [9, pp. 10], we have

$$\gamma_\infty = V_1 A_1, \quad \gamma_{1/3,r} = V_2 A_2 \begin{pmatrix} 1 & r+1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{1/2,s} = V_3 A_3 \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \gamma_{0,t} = V_4 A_4 \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

By (3.1), $F(V_\ell z) = F(z)$ for $\ell = 1, 2$ and $F(V_\ell z) = -F(z)$ for $\ell = 3, 6$. Hence, if $\zeta_6 := e(1/6)$ is a primitive sixth root of unity, then

$$\begin{aligned} F|_0\gamma_\infty(z) &= F(z) = e(-z) - 4 + \sum_{n=1}^{\infty} a(n)e(nz) \\ F|_0\gamma_{1/3,r}(z) = F\left(\frac{z+r+1}{2}\right) &= \zeta_6^{3-3r} e(-z/2) - 4 + \sum_{n=1}^{\infty} \zeta_6^{3n(r+1)} a(n)e(nz/2) \\ F|_0\gamma_{1/2,s}(z) = -F\left(\frac{z+s}{3}\right) &= \zeta_6^{3-2s} e(-z/3) + 4 + \sum_{n=1}^{\infty} \zeta_6^{3+2ns} a(n)e(nz/3) \\ F|_0\gamma_{0,t}(z) = -F\left(\frac{z+t}{6}\right) &= \zeta_6^{3-t} e(-z/6) + 4 + \sum_{n=1}^{\infty} \zeta_6^{3+nt} a(n)e(nz/6) \end{aligned}$$

Meanwhile, a calculation using the definition of $\mathcal{P}_F(z)$ and the group law on the Atkin-Lehner operators shows that

$$\mathcal{P}_F(W_\ell z) = \beta(\ell)\mathcal{P}_F(z), \tag{3.6}$$

and hence

$$\begin{aligned}
\mathcal{P}_F|_0\gamma_\infty(z) &= \mathcal{P}_F(z) = e(-z) + O(1)\\
\mathcal{P}_F|_0\gamma_{1/3,r}(z) &= \mathcal{P}_F\left(\frac{z+r+1}{2}\right) = \zeta_6^{3-3r}e(-z/2) + O(1)\\
\mathcal{P}_F|_0\gamma_{1/2,s}(z) &= -\mathcal{P}_F\left(\frac{z+s}{3}\right) = \zeta_6^{3-2s}e(-z/3) + O(1)\\
\mathcal{P}_F|_0\gamma_{0,t}(z) &= -\mathcal{P}_F\left(\frac{z+t}{6}\right) = \zeta_6^{3-t}e(-z/6) + O(1)
\end{aligned}$$

From the preceding computations we find that F and $\mathcal{P}_F$ have the same principal parts in the cusps of $\Gamma_0(6)$. Therefore, $F - \mathcal{P}_F$ is a bounded harmonic function on a compact Riemann surface, and hence constant. In particular, we have $F - \mathcal{P}_F = C_F$ for a constant C_F where

$$C_F = -4 - b_F(0) + \sum_{n=1}^{\infty} a(n)e(nz) + e(-\bar{z}) - \sum_{n=1}^{\infty} b_F(-n)e(-n\bar{z}) - \sum_{n=1}^{\infty} b_F(n)e(nz).$$

Take the limit of both sides as $\mathrm{Im}(z) \to \infty$ to get

$$C_F = -4 - b_F(0).$$

To compute $b_F(0)$, we begin as in [9, Lemma 3.1], utilizing

$$S(-\bar{\ell}, 0; c) = \mu(c)$$

to obtain

$$b_F(0) = 4\pi^2 \sum_{\ell|6} \frac{\beta(\ell)}{\ell} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,\ell)=1}} \frac{\mu(c)}{c^2}.$$

For each $\ell \mid 6$, the rightmost sum then reduces to

$$\sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,\ell)=1}} \frac{\mu(c)}{c^2} = \frac{\ell^2}{36} \sum_{\substack{d=1 \\ (d,\ell)=1}}^{\infty} \frac{\mu(6d/\ell)}{\ell^2} = \frac{1}{\zeta(2)} \begin{cases} 1/24 & \ell = 1 \\ -1/6 & \ell = 2 \\ -3/8 & \ell = 3 \\ 3/2 & \ell = 6. \end{cases}$$

The evaluation $\zeta(2) = \pi^2/6$ then grants

$$b_F(0) = 24\left(\frac{1}{24} - \frac{1}{12} + \frac{1}{8} - \frac{1}{4}\right) = -4.$$

It follows that $C_F = 0$ and hence $F(z) = \mathcal{P}_F(z)$. Thus by comparing the Fourier expansion of $F(z)$ and $\mathcal{P}_F(z)$, we obtain $a(n) = b_F(n)$ for every $n \geq 1$, $b_F(-1) = 1$, and $b_F(-n) = 0$ for every $n \geq 2$. $\square$

We conclude this section by giving an effective bound for the Fourier coefficients $a(n)$ for $n \geq 1$.

Lemma 3.3. *For $n \geq 1$,*

$$|a(n)| \leq Ce(4\pi\sqrt{n}),$$

where

$$C := 8\sqrt{6}\pi^{3/2} + 16\pi^2\zeta^2(3/2).$$

Proof. We utilize the proof of [9, Lemma 3.1], which bounds similar coefficients

$$a'(n) = 2\pi \sum_{\ell|6} \frac{\mu(\ell)}{\sqrt{\ell}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/\ell} \\ (c,\ell)=1}} \frac{S(-\tilde{\ell}, n; c)}{c} I_1\left(\frac{4\pi\sqrt{n}}{c\sqrt{\ell}}\right)$$

by $C\sqrt{n}e(4\pi\sqrt{n})$ for the given C; our result follows then from $|\mu(\ell)| = |\beta(\ell)| = 1$ for all $\ell \mid 6$ and multiplication by $n^{-1/2}$. □

4. Proof of Theorem 1.1

Given a form $Q \in \mathcal{Q}_\Delta$ and corresponding coset representative $\gamma_Q \in \mathbf{C}_6$, let $h_Q \in \{1, 2, 3, 6\}$ be the width of the cusp $\gamma_Q(\infty)$, and let ζ_Q and $\phi_{n,Q}$ be the sixth roots of unity defined as follows:

Table 1

cusp $\gamma_Q(\infty)$	$\infty \simeq 1/6$	$1/3$	$1/2$	$0 \simeq 1$
ζ_Q	1	ζ_6^{3-3r}	ζ_6^{3-2s}	ζ_6^{3-t}
$\phi_{n,Q}$	1	$\zeta_6^{3n(r+1)}$	ζ_6^{3+2ns}	ζ_6^{3+nt}

Then from the calculation in Proposition 3.2 we can write

$$F|_0\gamma_Q(z) = \zeta_Q e(-z/h_Q) - 4\beta(h_Q) + \sum_{n=1}^{\infty} \phi_{n,Q} a(n) e(nz/h_Q). \tag{4.1}$$

Now, recall the Brunier-Schwagenscheidt formula [1],

$$\alpha(n) = -\frac{1}{\sqrt{|D_n|}} \operatorname{Im}(S(n)). \tag{4.2}$$

We use this to give an effective bound on $S(n)$ and hence obtain our result for $\alpha(n)$. By (3.2) and (3.4),

$$\begin{aligned} S(n) &= \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u) S_u(n) \\ &= \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u) \sum_{[Q]\in \mathcal{Q}^{\text{prim}}_{D_n/u^2,6,1}/\Gamma_0(6)} F(\tau_Q) \\ &= \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u) \sum_{[Q]\in \mathcal{Q}_{D_n/u^2}} F|_0\gamma_Q(\tau_Q) \end{aligned}$$

which, by (4.1), yields

$$S(n)=\sum_{\substack{u>0\\u^2|D_n}}\varepsilon(u)\sum_{Q\in\mathcal{Q}_{D_n/u^2}}\zeta_Q e(-\tau_Q/h_Q)=\sum_{Q\in\mathcal{Q}_{D_n}}\zeta_Q e(-\tau_Q/h_Q)+E_1(n)+E_2(n)$$

where

$$E_1(n):=\sum_{\substack{u>1\\u^2|D_n}}\varepsilon(u)\sum_{Q\in\mathcal{Q}_{D_n/u^2}}\zeta_Q e(-\tau_Q/h_Q)$$

and

$$E_2(n):=4\beta(h_Q)\sum_{\substack{u>0\\u^2|D_n}}\varepsilon(u)h(D_n/u^2)+\sum_{n=1}^{\infty}a(n)\sum_{\substack{u>0\\u^2|D_n}}\varepsilon(u)\phi_{n,Q}e(n\tau_Q/h_Q).$$

To analyze the main term, note that for any $Q=[a_Q,b_Q,c_Q]\in\mathcal{Q}_{D_n/u^2}$, we have

$$a_Q h_Q\equiv 0 \pmod 6 \tag{4.3}$$

and

$$e(-\tau_Q/h_Q)=\zeta_{2a_Qh_Q}^{b_Q}\exp\left(\frac{\pi\sqrt{|D_n|/u^2}}{a_Qh_Q}\right) \tag{4.4}$$

We consider those forms $Q\in\mathcal{Q}_{D_n}$ with $a_Qh_Q=6$ and $a_Qh_Q=12$. We examine Table 2, which contains the value of c_Q for those forms $Q\in\mathcal{Q}^{\text{prim}}_{D_n,6,1}/\Gamma_0(6)$ with $1\le a_Q\le 12$.

Table 2

$a_Q\backslash b_Q$	±1	±3	±5	±7	±9	±11
1	$6n$					
2	$3n$					
3	$2n$					
4	$\frac{3n}{2}$	$\frac{3n+1}{2}$				
5	$\frac{6n}{5}$	$\frac{6n+2}{5}$				
6	n		$n+1$			
7	$\frac{6n}{7}$	$\frac{6n+2}{7}$	$\frac{6n+6}{7}$			
8	$\frac{3n}{4}$	$\frac{3n+1}{4}$	$\frac{3n+3}{4}$	$\frac{3n+6}{4}$		
9	$\frac{2n}{3}$		$\frac{2n+2}{3}$	$\frac{2n+4}{3}$		
10	$\frac{3n}{5}$	$\frac{3n+1}{5}$		$\frac{3n+6}{5}$	$\frac{3n+10}{5}$	
11	$\frac{6n}{11}$	$\frac{6n+2}{11}$	$\frac{6n+6}{11}$	$\frac{6n+12}{11}$	$\frac{6n+20}{11}$	
12	$\frac{n}{2}$		$\frac{n+1}{2}$	$\frac{n+2}{2}$		$\frac{n+5}{2}$

The forms with $a_Qh_Q=6$ are then, via [14, Table 1],

$$Q_1=[1,1,6n],\quad Q_2=[2,1,3n],\quad Q_3=[3,1,2n],\quad Q_4=[6,1,n]$$

with coset representatives

$$\gamma_{Q_1} = \gamma_{0,1}, \quad \gamma_{Q_2} = \gamma_{1/2,-1}, \quad \gamma_{Q_3} = \gamma_{1/3,0}, \quad \gamma_{Q_4} = \gamma_{\infty}.$$

Similarly, the forms with $a_Q h_Q = 12$ are

$$\begin{aligned} Q_5^0 &= [2,-1,3n] & Q_5^1 &= [2,-1,3n] \\ Q_6^0 &= [4,1,3n/2] & Q_6^1 &= [4,-3,(3n+1)/2] \\ Q_7^0 &= [6,-5,n+1] & Q_7^1 &= [6,-5,n+1] \\ Q_8^0 &= [12,1,n/2] & Q_8^1 &= [12,-11,(n+5)/2] \end{aligned}$$

with coset representatives

$$\begin{aligned} \gamma_{Q_5^0} &= \gamma_{0,0} & \gamma_{Q_5^1} &= \gamma_{0,3} \\ \gamma_{Q_6^0} &= \gamma_{\frac{1}{2},1} & \gamma_{Q_6^1} &= \gamma_{\frac{1}{2},2} \\ \gamma_{Q_7^0} &= \gamma_{\frac{1}{3},0} & \gamma_{Q_7^1} &= \gamma_{\frac{1}{3},1} \\ \gamma_{Q_8^0} &= \gamma_{\infty} & \gamma_{Q_8^1} &= \gamma_{\infty}. \end{aligned}$$

Thus, for $n \equiv r \pmod 2$, write

$$\sum_{Q \in \mathcal{Q}_{D_n}} \zeta_Q e(-\tau_Q/h_Q) = \sum_{i=1}^{4} \zeta_{Q_i} e(-\tau_{Q_i}/h_{Q_i}) + \sum_{i=5}^{8} \zeta_{Q_i^r} e(-\tau_{Q_i^r}/h_{Q_i^r}) + E_3(n)$$

where

$$E_3(n) := \sum_{\substack{Q \in \mathcal{Q}_{D_n} \\ a_Q h_Q \geq 18}} \zeta_Q e(-\tau_Q/h_Q).$$

For $i = 1,2,3,4$, we find via Table 1 the sixth roots of unity

$$\zeta_{Q_1} = \zeta_6^2, \quad \zeta_{Q_2} = \zeta_6^5, \quad \zeta_{Q_3} = \zeta_6^3, \quad \zeta_{Q_4} = 1$$

and, for $i = 5,6,7,8$,

$$\begin{aligned} \zeta_{Q_5^0} &= \zeta_6^3 & \zeta_{Q_5^1} &= \zeta_6^0 \\ \zeta_{Q_6^0} &= \zeta_6^1 & \zeta_{Q_6^1} &= \zeta_6^{-1} \\ \zeta_{Q_7^0} &= \zeta_6^3 & \zeta_{Q_7^1} &= \zeta_6^0 \\ \zeta_{Q_8^0} &= 1 & \zeta_{Q_8^1} &= 1. \end{aligned}$$

We then compute via (4.4)

$$\sum_{i=1}^{4} \zeta_{Q_i} e(-\tau_{Q_i}/h_{Q_i}) = \exp(\pi\sqrt{|D_n|/6}) \sum_{i=1}^{4} \zeta_{Q_i} \zeta_{12}^{b_{Q_i}}$$

where, since $b_{Q_i} = 1$ for $i = 1,2,3,4$,

$$\zeta_{12} \sum_{i=1}^{4} \zeta_{Q_i} = \zeta_{12}(\zeta_6^3 + \zeta_6^1 + \zeta_6^3 + 1) = 0.$$

Meanwhile, if n is even,

$$\sum_{i=5}^{8} \zeta_{Q_i^0} \zeta_{24}^{b_{Q_i^0}} = \zeta_{24}^{-1}\zeta_6^3 + \zeta_{24}\zeta_6 + \zeta_{24}^{-5}\zeta_6^3 + \zeta_{24} = i\sqrt{6}$$

and, if n is odd,

$$\sum_{i=5}^{8} \zeta_{Q_i^1} \zeta_{24}^{b_{Q_i^1}} = \zeta_{24}^{-1} + \zeta_{24}^{-3}\zeta_6^{-1} + \zeta_{24}^{-5} + \zeta_{24}^{-11} = -i\sqrt{6}$$

so that

$$S(n) = (-1)^n i\sqrt{6}\exp(\pi\sqrt{|D_n|}/12) + E_1(n) + E_2(n) + E_3(n).$$

Thus, by (4.2),

$$\alpha(n) = (-1)^{n+1}\frac{\sqrt{6}}{\sqrt{24n-1}}e^{l(n)/2} + \mathrm{Im}(E_1(n) + E_2(n) + E_3(n)).$$

We now bound each error term; since $u \geq 5$, then $ua_Qh_Q \geq 30$ and via (4.4),

$$\begin{aligned}|E_1(n)| &\leq \sum_{\substack{u>1\\ u^2|D_n}} \sum_{Q\in\mathcal{Q}_{D_n/u^2}} \exp(\pi\sqrt{|D_n|}/a_Qh_Q)\\ &\leq H(D_n)\exp(\pi\sqrt{|D_n|}/30).\end{aligned}$$

To bound $E_2(n)$, we proceed analogously to [9, pp. 14–15] to obtain, via Lemma 3.3,

$$\begin{aligned}|E_2(n)| &\leq 4H(D_n) + CH(D_n)\sum_{n=1}^{\infty}\exp(4\pi\sqrt{n} - \pi n/2\sqrt{3})\\ &\leq C_1H(D_n)\end{aligned}$$

where

$$C_1 := 4 + C[2.08\times 10^{20} + 426] < 2.47\times 10^{23}.$$

Finally,

$$\begin{aligned}|E_3(n)| &\leq \sum_{\substack{Q\in\mathcal{Q}_{D_n}\\ a_Qh_Q\geq 18}} \exp(\pi\sqrt{|D_n|}/a_Qh_Q)\\ &\leq h(D_n)\exp(\pi\sqrt{|D_n|}/18).\end{aligned}$$

Let $E(n) := \mathrm{Im}(E_1(n) + E_2(n) + E_3(n))$; this total error then satisfies

$$\begin{aligned}|E(n)| &\leq |E_1(n)| + |E_2(n)| + |E_3(n)|\\ &\leq H(D_n)\left[C_1 + \exp(\pi\sqrt{|D_n|}/30)\right] + h(D_n)\exp(\pi\sqrt{|D_n|}/18)\\ &< (2.48\times 10^{23})H(D_n)\exp(\pi\sqrt{|D_n|}/18).\end{aligned}$$

By the class number bound from [9, pp. 17], then,

$$|E(n)| < (4.30\times 10^{23})2^{q(n)}\left|D_n\right|^2\exp(\pi\sqrt{|D_n|}/18).$$

5. Corollaries to Theorem 1.1

We make use of the effective bound on $p(n)$ for all $n \geq 1$ from [9, Lemma 4.2]:

$$p(n) = \frac{2\sqrt{3}}{24n-1}\left(1 - \frac{1}{l(n)}\right)e^{l(n)} + E_p(n) \tag{5.1}$$

where $|E_p(n)| \leq (1313)e^{l(n)/2}$.

Corollary 5.1. *For $r = 0, 1$ and $n \geq 4$,*

$$N(r, 2; n) = M(n)e^{l(n)} + (-1)^r R_2(n),$$

where

$$M(n) := \frac{\sqrt{3}}{24n-1}\left(1 - \frac{1}{l(n)}\right)$$

and

$$|R_2(n)| \leq (8.17 \times 10^{30})e^{l(n)/2}$$

Proof. Utilizing (5.1) grants, via Theorem 1.1,

$$\begin{aligned} N(0, 2; n) &= \frac{p(n) + \alpha(n)}{2} \\ &= \frac{\sqrt{3}}{24n-1}\left(1 - \frac{1}{l(n)}\right)e^{l(n)} + R_2(n) \end{aligned}$$

and similarly

$$\begin{aligned} N(1, 2; n) &= \frac{p(n) - \alpha(n)}{2} \\ &= \frac{\sqrt{3}}{24n-1}\left(1 - \frac{1}{l(n)}\right)e^{l(n)} - R_2(n), \end{aligned}$$

where

$$R_2(n) := (-1)^{n-1}\frac{\sqrt{6}}{2\sqrt{24n-1}}e^{l(n)/2} + \frac{1}{2}(E_p(n) + E(n)).$$

We then have

$$\begin{aligned} |R_2(n)| &\leq \left(657 + \frac{\sqrt{6}}{2\sqrt{24n-1}}\right)e^{l(n)/2} + (2.15 \times 10^{23})2^{q(n)}\left|D_n\right|^2 e^{l(n)/3} \\ &\leq (8.17 \times 10^{30})e^{l(n)/2}. \end{aligned}$$

□

Corollary 5.2. *For all $n \geq 4$,*

$$\frac{N(r, 2; n)}{p(n)} = \frac{1}{2} + (-1)^r E_r(n),$$

where

$$|E_r(n)| \leq (1.89 \times 10^{32})e^{-l(n)/3}$$

Proof. Note that
$$\frac{N(r,2;n)}{p(n)} = \frac{1}{2} + (-1)^r \frac{\alpha(n)}{2p(n)}.$$
Let $E_r(n) := \alpha(n)/2p(n)$. We utilize a crude lower bound for $p(n)$ for $n \geq 4$
$$\frac{\sqrt{3}}{96} e^{l(n)} \leq \frac{\sqrt{3}}{12n}\left(1 - \frac{1}{\sqrt{n}}\right) e^{l(n)} < p(n)$$
due to Ono and Bessenrodt [4], and compute
$$\begin{aligned} |E_r(n)| &\leq \frac{48}{\sqrt{3}} e^{-l(n)} \left(\frac{\sqrt{6}}{\sqrt{24n-1}} e^{l(n)/2} + |E(n)| \right) \\ &\leq \frac{48\sqrt{2}}{\sqrt{24n-1}} e^{-l(n)/2} + (1.20 \times 10^{25}) 2^{q(n)} \left|D_n\right|^2 e^{-2l(n)/3} \\ &\leq (1.89 \times 10^{32}) e^{-l(n)/3}. \end{aligned}$$
□

6. Proof of Theorem 1.2

We first require the following lemma:

Lemma 6.1. *For $r = 0$ (resp. $r = 1$), we have that*
$$M(n)\left(1 - \frac{1}{\sqrt{n}}\right) e^{l(n)} < N(r,2;n) < M(n)\left(1 + \frac{1}{\sqrt{n}}\right) e^{l(n)}$$
for all $n \geq 8$ (resp. 7).

Proof. From Corollary 5.1, we have that
$$M(n)e^{l(n)} - |R_2(n)| < N(r,2;n) < M(n)e^{l(n)} + |R_2(n)|$$
with
$$|R_2(n)| \leq (8.17 \times 10^{30}) e^{l(n)/2}.$$
We then calculate that, for all $n \geq 4543$,
$$8.17 \times 10^{30} < \frac{M(n)}{\sqrt{n}} e^{l(n)/2}$$
and verify with a computer and the OEIS [16] the result for $n < 4543$. □

We now proceed with the full proof. Assume $11 \leq a \leq b$ and let $b = Ca$ where $C \geq 1$. By Lemma 6.1 we have the inequalities
$$N(r,2;a)N(r,2;Ca) > M(a)M(Ca)\left(1 - \frac{1}{\sqrt{a}}\right)\left(1 - \frac{1}{\sqrt{Ca}}\right) e^{l(a)+l(Ca)}$$
and
$$N(r,2;a+Ca) < M(a+Ca)\left(1 + \frac{1}{\sqrt{a+Ca}}\right) e^{l(a+Ca)}.$$
Thus, we seek conditions on $a > 1$ such that
$$e^{T_a(C)} > \frac{M(a+Ca)}{M(a)M(Ca)} S_a(C), \tag{6.1}$$

where

$$T_a(C) := l(a) + l(Ca) - l(a + Ca) \text{ and } S_a(C) := \frac{\left(1 + \frac{1}{\sqrt{a+Ca}}\right)}{\left(1 - \frac{1}{\sqrt{a}}\right)\left(1 - \frac{1}{\sqrt{Ca}}\right)}.$$

Taking logarithms in turn grants an equivalent formulation

$$T_a(C) > \log\left(\frac{M(a + Ca)}{M(a)M(Ca)}\right) + \log S_a(C). \tag{6.2}$$

Furthermore, as functions of C, T_a is strictly increasing and S_a strictly decreasing, so that it suffices to show that

$$T_a(1) > \log\left(\frac{M(a + Ca)}{M(a)M(Ca)}\right) + \log S_a(1)$$

for all $a \geq 8$, and, with $M(a + Ca)/M(Ca) \leq 1$ for all such a, we may show that

$$T_a(1) > \log S_a(1) - \log M(a). \tag{6.3}$$

Calculation of $T_a(1)$ and $S_a(1)$ shows that (6.3) holds for $a \geq 18$.

To complete the proof, assume that $11 \leq a \leq 17$. For each such integer a, we calculate the real number C_a for which

$$T_a(C_a) = \log S_a(C_a) - \log M(a).$$

The values C_a are listed in the table below.

Table 3

a	C_a	$\max b$
11	2.20...	24
12	1.86...	22
13	1.62...	21
14	1.43...	20
15	1.27...	19
16	1.15...	18
17	1.05...	17

By the discussion above, if $b = Ca$ is an integer for which $C > C_a$ holds, then (6.2) holds, which in turn grants the theorem in these cases. Only finitely many cases remain, namely the pairs integers where $11 \leq a \leq 17$ and $1 \leq b/a \leq C_a$. We compute $N(r, 2; a)$, $N(r, 2; b)$, and $N(r, 2; a + b)$ directly in these cases to complete the proof.

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