PATTERNS ARISING IN THE KERNEL OF GENERALIZED DEDEKIND SUMS

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ABSTRACT. We study a generalized version of the Dedekind sum associated with a pair of non-trivial primitive Dirichlet characters χ_1 and χ_2, denoted S_{χ_1,χ_2}. We investigate the kernel of the crossed homomorphism $S_{\chi_1,\chi_2} : \Gamma_0(q_1q_2) \to \mathbb{C}$ by expanding the work of Dillon and Gaston who showed that S_{χ_1,χ_2} is nontrivial. We apply the work of Stucker, Vennos, and Young to show that the kernel of S_{χ_1,χ_2} is strongly nontrivial. We also show properties of the kernel that lead to interesting symmetries graphically and examine the kernel's relationship with the commutator subgroup of $\Gamma_0(q_1q_2)$.

1. Introduction

1.1. **Background and prior work.** Dedekind sums were first introduced by Richard Dedekind as a way to express the transformation formula satisfied by the Dedekind η-function. Throughout the years, the study of Dedekind sums have shown up in different areas of mathematics including algebraic number theory and combinatorial geometry. In this paper we study a generalization of the Dedekind sum associated to a pair of Dirichlet characters studied in [5] and [3]. Let h, k be coprime integers with $k \geq 1$. The classical Dedekind sum is defined by:

$$s(h,k) = \sum_{n \bmod k} B_1\left(\frac{n}{k}\right) B_1\left(\frac{hn}{k}\right), \tag{1}$$

where B_1 denotes the first Bernoulli function

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases} \tag{2}$$

For more background on the Dedekind η-function and the classical Dedekind sum, we refer the reader to [1].

Many different generalized versions of the Dedekind sum have appeared in the literature. In this paper, we further study the newform Dedekind sum studied recently by Stucker, Vennos, and Young in [5]; see their introduction for a more thorough historical survey of previous work on these types of Dedekind sums. Throughout we let χ_1 and χ_2 be primitive Dirichlet characters modulo q_1 and q_2, respectively, with q_1, $q_2 > 1$. We let $\Gamma_0(q_1q_2)$ denote the congruence subgroup of level q_1q_2. We take Theorem 1.2 from [5] as a definition of the newform Dedekind sum associated to a pair of Dirichlet characters, restated here:

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$ with $c > 0$ and $\chi_1\chi_2(-1) = 1$, define the newform Dedekind sum as

$$S_{\chi_1,\chi_2}(\gamma) = \sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right). \tag{3}$$

For simplicity, we will often refer to the newform Dedekind sum simply as the Dedekind sum. Since the $S_{\chi_1,\chi_2}(\gamma)$ in (3) only depends on the first column of γ, we will often write $S_{\chi_1,\chi_2}(a,c)$ in place of $S_{\chi_1,\chi_2}(\gamma)$.

Let $\psi = \chi_1\overline{\chi_2}$, and for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$, we define $\psi(\gamma) = \chi_1\overline{\chi_2}(d)$. The most important basic property of the Dedekind sum $S_{\chi_1,\chi_2} : \Gamma_0(q_1q_2) \to \mathbb{C}$ is that it is a crossed homomorphism, which we record with the following:

Theorem 1.1. *For all $\gamma_1, \gamma_2 \in \Gamma_0(q_1q_2)$, we have*

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)S_{\chi_1,\chi_2}(\gamma_2). \tag{4}$$

A proof can be found in [5, Lemma 2.2]. We remark that [5, Definition 1.1] started with a different definition of the Dedekind sum (via a kind of generalized Kronecker limit formula for the newform Eisenstein series E_{χ_1,χ_2}) from which (4) was easy to show (in contrast to (3)). After some additional work, they derived the finite formula (3), which is only valid for $c > 0$. However, it is easy to see from the original definition that for $c = 0$ we have $S_{\chi_1,\chi_2}(1, 0) = 0$, and that $S_{\chi_1,\chi_2}(-a, -c) = S_{\chi_1,\chi_2}(a, c)$, which takes care of $c < 0$.

Remark 1.2. *For $\gamma \in \Gamma_1(q_1q_2)$, then $\psi(\gamma) = 1$. Therefore, the crossed homomorphism S_{χ_1,χ_2} becomes a group homomorphism from $\Gamma_1(q_1q_2)$ into $(\mathbb{C}, +)$.*

The reciprocity formula for the classical Dedekind sum is one of its most interesting and important features. The following reciprocity formula for S_{χ_1,χ_2} is proved in [5] via the action of the Fricke involution $\omega = \left(\begin{smallmatrix} 0 & -1 \\ q_1q_2 & 0 \end{smallmatrix}\right)$.

Theorem 1.3 (Reciprocity formula [5])**.** *For $\gamma = \left(\begin{smallmatrix} a & b \\ cq_1q_2 & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$, let $\gamma' = \left(\begin{smallmatrix} d & -c \\ -bq_1q_2 & a \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$. If χ_1, χ_2 are even, then*

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_2,\chi_1}(\gamma'). \tag{5}$$

If χ_1, χ_2 are odd, then with $\tau(\chi)$ denoting the standard Gauss sum, we have

$$S_{\chi_1,\chi_2}(\gamma) = -S_{\chi_2,\chi_1}(\gamma') + (1 - \psi(\gamma)) \left(\frac{\tau(\overline{\chi}_1)\tau(\overline{\chi}_2)}{(\pi i)^2} \right) L(1, \chi_1)L(1, \chi_2). \tag{6}$$

Our main interest in this paper is to understand the structure of the kernels of the Dedekind sums. To make our objects of interest more precise, we make the following definition.

Definition 1.4. Let χ_1 and χ_2 be non-trivial primitive Dirichlet characters modulo q_1 and q_2, respectively, with $q_1, q_2 > 1$. Then we denote the kernel associated to χ_1, χ_2, by

$$K_{\chi_1,\chi_2} = \ker(S_{\chi_1,\chi_2}) = \{\gamma \in \Gamma_0(q_1q_2) : S_{\chi_1,\chi_2}(\gamma) = 0\}.$$

We let $K^1_{\chi_1,\chi_2}$ denote $K_{\chi_1,\chi_2} \cap \Gamma_1(q_1q_2)$. Moreover, we define

$$K_{q_1,q_2} = \bigcap_{\substack{\chi_1,\chi_2 \\ \chi_1\chi_2(-1)=1}} K_{\chi_1,\chi_2},$$

where χ_i runs over primitive characters modulo q_i, $i = 1, 2$. We similarly let $K^1_{q_1,q_2} = K_{q_1,q_2} \cap \Gamma_1(q_1q_2)$.

Remark 1.5. *One can similarly consider $K_{\chi_1,\chi_2} \cap \Gamma(q_1q_2)$, but since S_{χ_1,χ_2} depends only on the first column of γ, this is essentially the same as $K^1_{\chi_1,\chi_2}$. More precisely, what we mean here is that $\Gamma_1(q_1q_2) = \bigcup_b \Gamma(q_1q_2)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ is trivially in the kernel of any Dedekind sum.*

The following theorem of Dillon and Gaston [3] shows that S_{χ_1,χ_2} is non-trivial in a strong sense:

Theorem 1.6 (Strong nontriviality [3])**.** *For each $c > 0$ such that $q_1q_2|c$, there exists an a so that $S_{\chi_1,\chi_2}(a, c) \neq 0$.*

Remark 1.7. *One way to interpret this result of Dillon and Gaston is that it shows that K_{χ_1,χ_2} is not "too big" (keeping account of the size of c, the lower-left entry of elements of $\Gamma_0(q_1q_2)$).*

Our first main result shows that $K^1_{q_1,q_2}$ is not "too small."

Theorem 1.8 (Kernel is strongly nontrivial). *For every $c \in \mathbb{Z}$, there exists $\gamma = \begin{pmatrix} a & b \\ cq_1q_2 & d \end{pmatrix} \in \Gamma(q_1q_2)$ such that $\gamma \in K^1_{q_1,q_2}$.*

Next we discuss some relationships between commutator subgroups and kernels of the newform Dedekind sums. We begin with a general discussion. If G is a group, we let $[G,G]$ denote its commutator subgroup (i.e., the smallest subgroup of G containing all commutators $xyx^{-1}y^{-1}$ with $x, y \in G$). It is well-known (and easy to check) that if $\varphi : G \to H$ is a group homomorphism, with H abelian, then $[G,G] \subseteq \ker(\varphi)$. We also recall that the abelianization of a group, denoted G^{ab} is defined by $G^{\mathrm{ab}} = [G,G]\backslash G$. It is known that the abelianization of $G = SL_2(\mathbb{Z})$ is $\mathbb{Z}/12\mathbb{Z}$, which implies that there are no non-trivial group homomorphisms from $SL_2(\mathbb{Z})$ to $\mathbb{C}$. Theorem 1.6 is in sharp contrast to the level 1 case.

One naturally is led to wonder to what extent the commutator subgroups of $\Gamma_0(q_1q_2)$, $\Gamma_1(q_1q_2)$, etc. account for the kernels of the Dedekind sums. Our second main result shows that $K^1_{q_1,q_2}$ is much larger than the commutator subgroup of $\Gamma(q_1q_2)$ (cf. Remark 1.5).

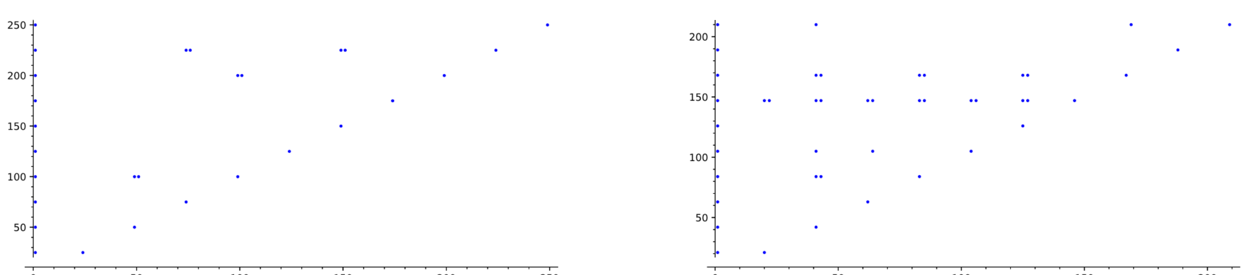
Theorem 1.9. *We have $[\Gamma(q_1q_2), \Gamma(q_1q_2)] \subsetneq K^1_{q_1,q_2}$.*

Remark 1.10. *In fact, we show in Proposition 3.1 below that $[\Gamma(q_1q_2), \Gamma(q_1q_2)] \subseteq \Gamma(q_1^2q_2^2)$. In contrast, Theorem 1.8 produces elements that are clearly not in $\Gamma(q_1^2q_2^2)$ (indeed, there is no restriction on the lower-left entry besides divisibility by q_1q_2). This explains why we stated that $K^1_{q_1,q_2}$ is much larger than the commutator subgroup.*

Our final main observation is that there exists a natural Galois action on the Dedekind sums, which can easily be read off from (3). This is discussed in Section 4.

2. Numerical data and proof of Theorem 1.8

We begin this section with some numerical calculations of K_{q_1,q_2}. We let (a,c) represent the left column of $\gamma \in \Gamma_0(q_1q_2)$. By (3), it can be shown that $S_{\chi_1,\chi_2}(a,c) = S_{\chi_1,\chi_2}(b,c)$ where $a \equiv b \bmod c$. Therefore, we only need to examine the pairs (a,c) such that $a \in \{1, ..., c-1\}$. Using SageMath [6], for all primes $3 \le q_1, q_2 \le 11$, we computed the elements of $K_{\chi_1,\chi_2}, K^1_{\chi_1,\chi_2}, K_{q_1,q_2}$, and $K^1_{q_1,q_2}$ with $1 \le c \le 10q_1q_2$, directly using the finite sum formula (3) as our definition. Consider the example in Figure 1a where $q_1 = q_2 = 5$ in which we display the elements of K_{q_1,q_2} for $1 \le c \le 250$.



(a) $K_{5,5}$ (b) $K_{7,3}$

Figure 1. K_{q_1,q_2} for $1 \le c \le 10q_1q_2$

From Figure 1a, Figure 1b, and other similar graphs, we found the vertical line formed when $a = 1$ to consistently appear. We prove this in Corollary 2.4. We also found other lines corresponding to similar patterns shown in the following propositions.

Proposition 2.1. *Let χ_1 and χ_2 be non-trivial primitive Dirichlet characters modulo q_1 and q_2, respectively, with $q_1, q_2 > 1$. Then $S_{\chi_1,\chi_2}(1, q_1q_2) = 0$.*

Proof. We take $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ q_1q_2 & 1 \end{smallmatrix}\right)$ in Theorem 1.3, so $\gamma' = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. It is easy to see that $S_{\chi_1,\chi_2}(\gamma') = 0$, and that $1 - \psi(\gamma) = 0$. Therefore, Proposition 2.1 follows from the reciprocity formula. □

In [3], Dillon and Gaston showed that for $c \geq 1$ and $q_1q_2|c$,

$$S_{\chi_1,\chi_2}(-a, c) = -\chi_2(-1)S_{\chi_1,\chi_2}(a, c). \tag{7}$$

Letting $c = q_1q_2$ and $a = 1$ in (7), we can use Proposition 2.1, to conclude that $S_{\chi_1,\chi_2}(q_1q_2 - 1, q_1q_2) = 0$. Moreover, using (7), one can easily observe that for $a \pmod c$, if $S_{\chi_1,\chi_2}(a, c) = 0$ then $S_{\chi_1,\chi_2}(c - a, c) = 0$. This symmetry between the pairs (a, c) and $(c - a, c)$ can be seen in Figure 1 above.

Proposition 2.2. *Let χ_1 and χ_2 be non-trivial primitive Dirichlet characters modulo q_1 and q_2, respectively, with $q_1, q_2 > 1$. Then if $S_{\chi_1,\chi_2}(1 + ndq_1q_2, d^2q_1q_2) = 0$ for some $n, d \in \mathbb{Z}$, then $S_{\chi_1,\chi_2}(1 + kndq_1q_2, kd^2q_1q_2) = 0$ for all $k \in \mathbb{Z}$.*

Remark 2.3. *Proposition 2.2 can be used to explain some linear patterns visible among the points in Figure 1. For instance, we have $a = 51$, $c = 100$ in $K_{5,5}$ visible in Figure1a, which corresponds to $d = 2$, $n = 1$, in Proposition 2.2. The point $a = 101$, $c = 200$ corresponds to $k = 2$ in Proposition 2.2.*

Proof. We prove this by showing an interesting property of matrices of the form

$$\gamma = \begin{pmatrix} 1 + ndq_1q_2 & -n^2q_1q_2 \\ d^2q_1q_2 & 1 - ndq_1q_2 \end{pmatrix} = I + QA, \quad \text{where} \quad A = \begin{pmatrix} nd & -n^2 \\ d^2 & -nd \end{pmatrix}, \quad Q = q_1q_2.$$

Note that $A^2 = 0$. Using this, one can easily show that $\gamma^k = I + kQA$, for any $k \in \mathbb{Z}$. Therefore, if $\gamma \in K_{\chi_1,\chi_2}$, then so is γ^k, which translates to the desired statement. □

Corollary 2.4. *Let χ_1 and χ_2 be primitive Dirichlet characters modulo q_1 and q_2, respectively, such that $q_1, q_2 > 1$ and $\chi_1\chi_2(-1) = 1$. Then $S_{\chi_1,\chi_2}(1, kq_1q_2) = 0$ for all $k \in \mathbb{Z}$.*

Proof. We apply Propositions 2.1 and 2.2, with $n = 0$ and $d = 1$. □

The points $(1, c)$ create a vertical line, depicted in Figure 1 above. Similarly, one can use (7) to conclude that the points $(c - 1, c)$ create a line of slope 1, also depicted in Figure 1.

Now we prove Theorem 1.8. By Corollary 2.4, we have that $\left(\begin{smallmatrix} 1 & 0 \\ cq_1q_2 & 1 \end{smallmatrix}\right) \in K^1_{\chi_1,\chi_2}$, for all choices of χ_1, χ_2, so the result follows immediately. □

Remark 2.5. *Figure 1 indicates that there exist examples of q_1, q_2 and c for which the only element $(a, c) \in K^1_{q_1,q_2}$ with $0 < a < c$ is the point exhibited in Corollary 2.4. In this sense, Corollary 2.4 is sharp.*

3. The commutator subgroup: Proof of Theorem 1.9

For any prime p, let $\Gamma(n; p)$ denote the principal congruence subgroup of $SL_n(\mathbb{Z})$ of level p. In [4], Lee and Szczarba show that the commutator subgroup $[\Gamma(n; p), \Gamma(n; p)] = \Gamma(n; p^2)$ for $n \geq 3$ and all primes p. In the following proposition, we adapt the proof of Lee and Szczarba to show a one-sided containment of the commutator subgroup for $n = 2$ and p not necessarily prime.

Proposition 3.1. *For any $Q \geq 1$, then $[\Gamma(Q), \Gamma(Q)] \subseteq \Gamma(Q^2)$.*

Proof. Define the map $\varphi : \Gamma(Q) \to M_{2\times2}(\mathbb{Z}/Q\mathbb{Z})$ by $\varphi(A) = \frac{A-I}{Q} \pmod Q$. It is not difficult to show φ is a group homomorphism. Since $M_{2\times2}(\mathbb{Z}/Q\mathbb{Z})$ is abelian, then $[\Gamma(Q), \Gamma(Q)] \subseteq \ker(\varphi)$. We can see that $\ker(\varphi) = \Gamma(Q^2)$ by the definition of φ. □

Remark 3.2. *Lee and Szczarba additionally show that the image of φ is the subset of $M_{2\times 2}(\mathbb{Z}/Q\mathbb{Z})$ of trace $\equiv 0 \mod Q$.*

Proposition 3.1 shows that $[\Gamma(q_1q_2), \Gamma(q_1q_2)] \subseteq \Gamma(q_1^2q_2^2)$. From Proposition 2.2 and Corollary 2.4, we see that the inclusion is strict, i.e. $\Gamma(q_1^2q_2^2) \subsetneq K^1_{q_1,q_2}$.

4. The Galois action

We now study the kernels further by comparing K_{χ_1,χ_2} for different choices of χ_1, χ_2, with a specified choice of q_1, q_2. Let $\zeta_n = e^{2\pi i/n}$. Figure 2 depicts $K^1_{\chi_1,\chi_2}$ for the Dedekind sum associated to $\chi_1 \mod 5$, the character mapping $2 \mapsto i$, and $\chi_2 \mod 11$, which maps $2 \mapsto \zeta_{10}$, for $0 < c \leq 1100$. However, we found that Figure 2 also represents the kernel for other characters, such as the pair $\chi_1' \bmod 5 : 2 \mapsto -i$ and $\chi_2' \bmod 11 : 2 \mapsto \zeta_{10}^3$.

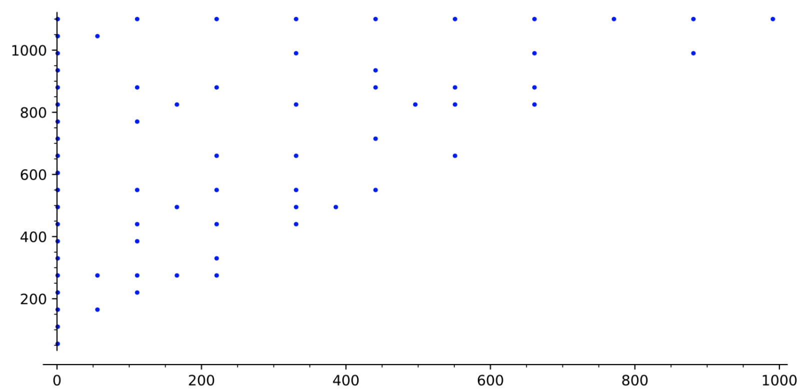


Figure 2. $K^1_{\chi_1,\chi_2}$ for $\chi_1 \bmod 5$ and $\chi_2 \bmod 11$

This reoccurring pattern of identical kernels was found for other conductors q_1, q_2 as well. To explain this pattern we turn to Galois theory. For two characters $\chi_1, \chi_2 \mod q_1$ and q_2 respectively, we let $F = \mathbb{Q}\left(\zeta_{\phi(q_1)}, \zeta_{\phi(q_2)}\right) = \mathbb{Q}(\zeta_{[\phi(q_1),\phi(q_2)]})$ be the rational field extension over $\mathbb{Q}$ containing the $\phi(q_1)$ and $\phi(q_2)$-th primitive roots of unity. Let

$$\mathrm{Ded}(q_1, q_2) = \{S_{\chi_1,\chi_2} \mid \chi_1\chi_2(-1) = 1 \text{ and } \chi_i \text{ primitive modulo } q_i, i = 1, 2\}. \tag{8}$$

For $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, and χ a Dirichlet character taking values in F, let χ^σ denote the character defined by $n \to \sigma(\chi(n))$. By the definition of the Dedekind sum in (3), we see that $S_{\chi_1,\chi_2}(\gamma)$ lies in F, for all $\gamma \in \Gamma_0(q_1q_2)$, since the values taken by the Bernoulli function B_1 in (3) are rational.

Proposition 4.1. *Let $F = \mathbb{Q}\left(\zeta_{\phi(q_1)}, \zeta_{\phi(q_2)}\right)$. Then there exists a natural group action of* $\mathrm{Gal}(F/\mathbb{Q})$ *on* $\mathrm{Ded}(q_1, q_2)$ *which we denote as $S^\sigma_{\chi_1,\chi_2}$ for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$.*

Proof. We define the natural Galois action by $S^\sigma_{\chi_1,\chi_2}(\gamma) := \sigma(S_{\chi_1,\chi_2}(\gamma))$, which by the definition (3) equals $S_{\chi_1^\sigma,\chi_2^\sigma}(\gamma)$, for any $\gamma \in \Gamma_0(q_1q_2)$. From this definition it is easy to see that this is a group action. $\square$

Remark 4.2. *If $k \in \mathbb{Z}$ is coprime to $\phi(q_1)\phi(q_2)$, the mapping $\omega \to \omega^k$, where ω is a primitive root of unity in F, is an automorphism of F. In fact, all automorphisms in the* $\mathrm{Gal}(F/\mathbb{Q})$ *can be formed this way.*

Corollary 4.3. *If two Dedekind sums, S_{χ_1,χ_2} and $S_{\chi_1',\chi_2'}$, are in the same orbit of* $\mathrm{Ded}(q_1, q_2)$ *under the action of* $\mathrm{Gal}(F/\mathbb{Q})$*, then S_{χ_1,χ_2} and $S_{\chi_1',\chi_2'}$ have the same kernel.*

Remark 4.4. *Corollary 4.3 implies that when studying the kernel of Dedekinds sums associated to specified* q_1, q_2*, we only need to examine a representative for each orbit, which leads to a significant efficiency in computation.*

Proof. The statement that S_{χ_1,χ_2} and $S_{\chi_1',\chi_2'}$ lie in the same orbit simply means that there exists $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ so that $\chi_1' = \chi_1^\sigma$ and $\chi_2' = \chi_2^\sigma$. It is clear that if $\gamma \in K_{\chi_1,\chi_2}$, then $\gamma \in K_{\chi_1^\sigma,\chi_2^\sigma}$ also. □

Considering the example in Figure 2, letting $k = 3$ we see that

- $q_1 = 5 : \chi_1(2)^3 = i^3 = -i = \chi_1'(2)$
- $q_2 = 11 : \chi_2(2)^3 = \zeta_{10}^3 = \chi_2'(2)$.

It then follows that the Dedekind sums associated to the two pairs of characters are in the same orbit and subsequently have the same kernel, corroborating our corollary.

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