Polynomial Systems Supported on Circuits

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Goals

- Understand the behaviour of polynomial circuit system.
- Develop tools to analyze and characterize circuit systems
- Help develop algorithms to quickly solve circuit systems.

Introduction

- Consider the problem of solving (in $\mathbb{R}$) a sparse polynomial system: an $n \times n$ t-nominal system (where n and t are positive integers).
- The case of $t \leq n+1$ is generally well understood.
- The case of $t \geq n+3$ is difficult to work with.
- The case of $t = n+2$ is an ongoing problem

We say that any $n \times n$ $(n+2)$-nominal system is supported on a circuit (as long as it fufills some non-degeneracy requirements).

Past Literature

- Researchers have known since at least 2006 that solving polynomial circuit systems reduces to finding solutions to the univariate logarithmic form

$$\Lambda(x) = b_1 \log(\gamma_{1,1}x + \gamma_{1,0}) + \ldots + b_{n+1} \log(\gamma_{n+1,1}x + \gamma_{n+1,0})$$

- Rojas has performed considerable work in both analyzing the behaviour of these logarithmic sums and developing algorithms to count their roots (Rojas 1).

Motivation

- Knowledge of root spacing informs better usage of Newton's method.
- Newton's method is nearly optimal for zero finding.
- By Rolle's Theorem, roots and critical values are interlaced.

Experimental Approach

- Our experimentation was performed through MATLAB code.
- Through experimentation we have analyzed, among other items, the following quantities:
  - Coefficients of univariate reduction.
  - Magnitude of critical values.
  - Spacing of critical values.
  - Spacing of roots.
- Each 'data point' present was obtained by taking the arithmetic mean of various 'trials' in which the initial coefficients of the system were uniform random integers.

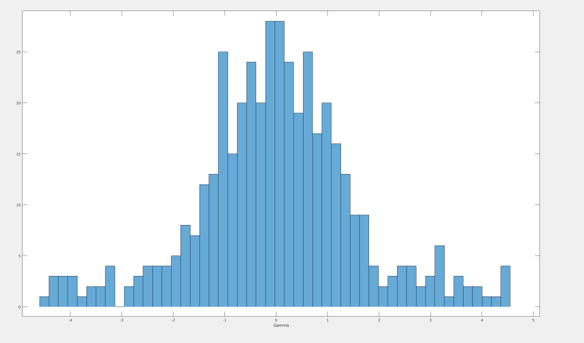
Analytical Approach

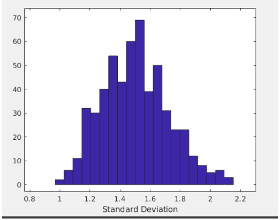
- Through a canonical form, we can restrict our attention to "tetrahedral circuits" whose exponent matrices are of the form:

$$A_d(\mathbf{v}) = \begin{bmatrix} d & 0 & \dots & 0 & v_1 & 0 \\ 0 & d & \dots & 0 & v_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & d & v_n & 0 \end{bmatrix}$$

- When analyzing how quantities change as a function of d, we will often employ asymptotic arguments.
- When applicable, we will also employ the Shapiro-Wilk test, a statistical test to determine normality (Shapiro and Wilk 591).

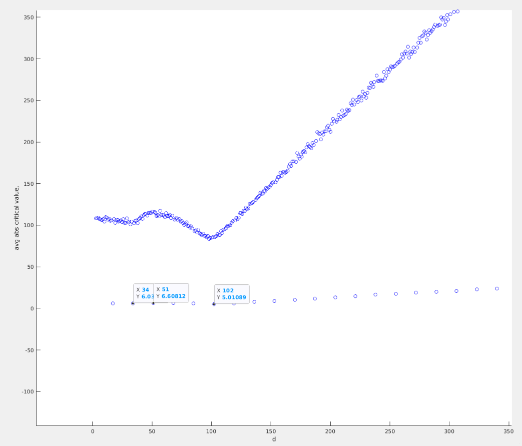
Coefficients





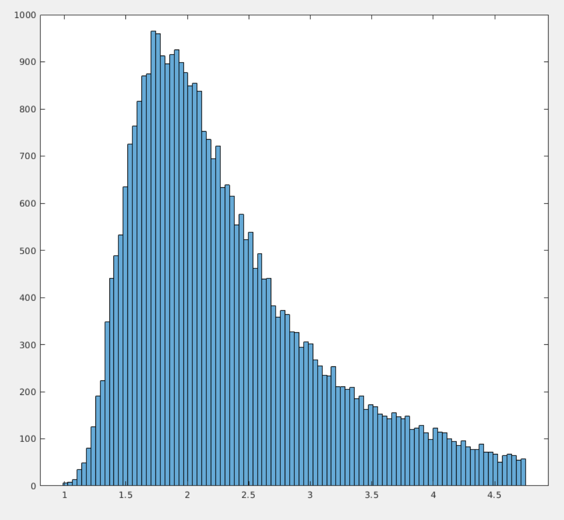
Parameters (first): $HMAX = 10,000$,
(second): $HMAX = 550$,

Critical Values

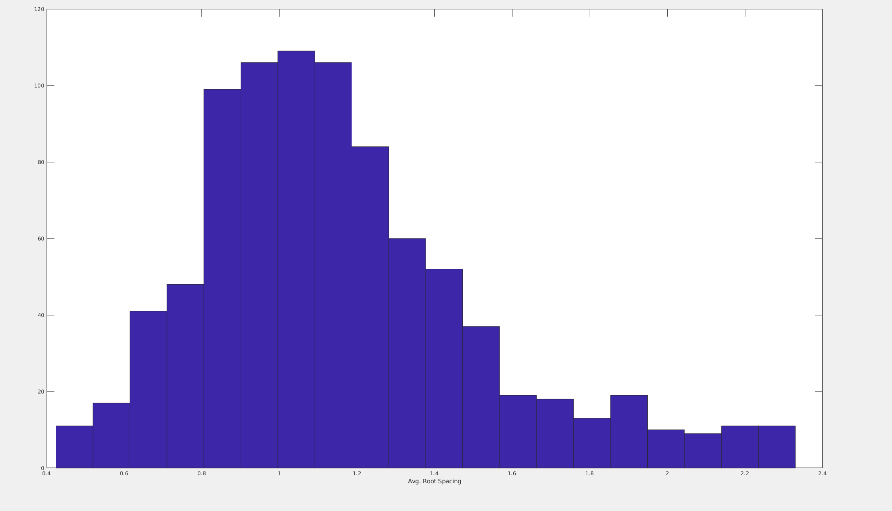


Parameters: $\mathbf{v} = [54, 31, 17]$

Critical Spacing



Root Spacing



Further Work

The following processes are ongoing and will hopefully be finalized in the months following the program's conclusion.

- Determine the role of the discriminant.
- Properly characterize discovered probability distributions.
- Further formalize discovered patterns and prove more conjectures.

References

Ali, Owais. "Gaussian Elimination." MathWorks, 4 May 2022, www.mathworks.com/ matlabcentral/fileexchange/109590-gaussian-elimination?s$_tid = prof_contriblnk$.

BenSada, Ahmed. "Shapiro-Wilk and Shapiro-Francia normality tests." MATLAB Central File Exchange, June 2014, www.mathworks.com/matlabcentral/fileexchange/13964-shapiro-wilk-and-shapiro-francia-normality-tests. Accessed 17 July 2022.

Bertrand, Benoît, et al. "Polynomial Systems with Few Real Zeroes." Mathematische Zeitschrift, vol. 253, no. 2, 23 Feb. 2006, pp. 361-85.Springer Link, https://doi.org/10.1007/s00209-005-0912-8. Accessed 9 June 2022.

Malajovich-Munoz, Gregorio. *On the Complexity of Path-Following Newton Algorithms for Solving Systems of Polynomial Equations with Integer Coefficients.* 1993. U of California at Berkeley, PhD thesis. *ProQuest Dissertations and Theses*, www.proquest.com/pqdtglobal/docview/304077744/BAC7DA1D26E14E8FPQ/1?accountid=7082.

Rojas, J. Maurice. "Counting Real Roots in Polynomial-Time for Systems Supported on Circuits." 9 Dec. 2020. *arXiv*, https://doi.org/10.48550/arXiv.2012.04868.

Shapiro, S. S., and M. B. Wilk. "An Analysis of Variance Test for Normality." Biometrika, vol. 52, no. 3/4, Dec. 1965, p. 591. JSTOR, https://doi.org/10.2307/2333709.