Visualizing $\mathcal{A}$-Discriminant Varieties and their Tropicalizations

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July 23, 2014

Agenda

- Problem
- Approaches
- Important Concepts
- Amoeba
- Approximation of the Amoeba
- Goals
- Applications

Problem: Solving Polynomial Equations

- Abel's Theorem states that, for polynomials of degree 5 or higher, it is not possible to express the general solutions of a polynomial equation in terms of radicals.
- This theorem points to the need for more general iterative algorithms that go beyond taking radicals.

Approaches: Sturm Sequences

Given $f(x) = x^4 - 2x^2 + 1$:

- $P_f(x) = (x^4 - 2x^2 + 1, 4x^3 - 4x, x^2 - 1, 0, 0)$
- $\sigma(P_f(-3)) = (1, -1, 1)$ and $\sigma(P_f(3)) = (1, 1, 1)$
- $V_f(-3) = 2$ and $V_f(3) = 0$

For f, the number of roots between -3 and 3 is 2.

When computing the Sturm Sequence for $f(x) = x^{317811} - 2x^{196418} + 1$, the polynomials needed to complete the computation have hundreds of thousands of digits.

Approaches: Classifying Polynomials

Two ways to classify the polynomial:

$$f(x, y) = c_0x^3 + c_1x^2y^2 + c_2y^3 + c_3$$

- Based on degree: $f(x, y)$ is a cubic polynomial.
- Based on number of variables and terms $f(x, y)$ is a bi-variate, 4 - nomial.

Using the second method can be useful when dealing with polynomials of high degree with few terms.

Approaches: Studying n Variate k-Nomials

For each $(n+k)$-nomial case, we have families of polynomials with the same exponents.

Example: $n+3$ Case

$$f(x) = c_0x^3 + c_1x^2 + c_2x + c_3$$

$$g(x,y) = c_0x^6y^2 + c_1x^2y^{-7} + c_2x^2y^5 + c_3x + c_4y$$

$7x^3 + 1x^2 + -4x + 8$ and $-23543x^3 + 12345x^2$ are in the same family.

Important Concepts: Support

- For each $(n+k)$-nomial case, we have families of polynomials with the same exponents.
- Each family can be represented by its support.

Definition

Given $f(x_1, x_2, ..., x_n) = c_1 x^{a_1} + c_2 x^{a_2} + \cdots + c_t x^{a_t}$ where t represents the number of terms, $c_i \in \mathbb{C}$, $a_i \in \mathbb{Z}^n$

$$supp(f) = \mathcal{A} = \{a_1, \ldots, a_t\}$$

Example: $n+3$ Case

$f(x) = c_0 x^3 + c_1 x^2 + c_2 x + c_3$

$g(x, y) = c_0 x^6 y^2 + c_1 x^2 y^{-7} + c_2 x^2 y^5 + c_3 x + c_4 y$

$$supp(f) = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \quad supp(g) = \begin{bmatrix} 6 & 2 & 2 & 1 & 0 \\ 2 & -7 & 5 & 0 & 1 \end{bmatrix}$$

Important Concepts: $\Delta_{\mathcal{A}}$ and $\nabla_{\mathcal{A}}$

- For a given support, we can find the $\mathcal{A}$-discriminant, $\Delta_{\mathcal{A}}$.
- $\nabla_{\mathcal{A}}$ refers to the zero set of $\Delta_{\mathcal{A}}$.
- Each element in $\nabla_{\mathcal{A}}$ represents a polynomial with degenerate roots (a root where the Jacobian determinant vanishes).

Example

Given $c_0x^2 + c_1x + c_2$

$\mathcal{A} = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$

$\Delta_{\mathcal{A}} = c_1^2 - 4c_0c_2$

$\nabla_{\mathcal{A}}$ refers to the solution set of $c_1^2 - 4c_0c_2$

Because $(2, 4, 2)$ and $(1, 6, 9)$ are elements of $\nabla_{\mathcal{A}}$, we know

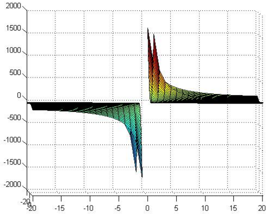
$$2x^2 + 4x + 2 \quad \text{and} \quad x^2 + 6x + 9$$

have degenerate roots.

Important Concepts: $\mathcal{A}$ - Discriminants

- We can plot $\nabla_{\mathcal{A}}$ in a dimension equal to the number of terms.
- The visualization represents every real polynomial in a family.
- Each point on the plot is a polynomial with degenerate roots.

Figure: Quadratic Case



Important Concepts: Parametrization

- The $\mathcal{A}$-discriminant polynomial can become difficult to calculate.
- We can find a parametrization to describe the solution set without solving for $\Delta_{\mathcal{A}}$.
- By taking the log of this parametrization, we obtain a visualization for understanding a family of polynomials, the amoeba.

Amoeba

- The amoeba of any polynomial, f, is the log of the absolute value of the zero set of f.
- To plot the $\mathcal{A}$-discriminant amoeba, we find the zero set, $\nabla_{\mathcal{A}}$, and plot $\log|\nabla_{\mathcal{A}}|$.
- We can create a visualization in a lower dimension by plotting the amoeba of the reduction of the polynomial.

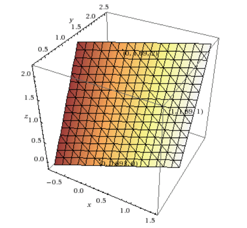
Amoeba: Reduced $\mathcal{A}$-Discriminant Amoeba

With division and rescaling $f(x) = c_0x^2 + c_1x + c_2$ can be reduced to $x^2 + x + c$.

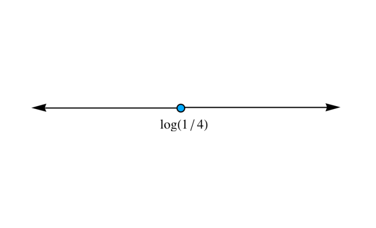
$$\Delta_{\mathcal{A}} = c_1^2 - 4c_0c_2$$

$$\overline{\Delta}_{\mathcal{A}} = 1 - 4c$$

Amoeba(Δ_A)



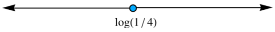
Amoeba($\overline{\Delta}_A$)



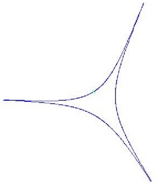
Amoeba: Visualization

We can visualize the reduced $\mathcal{A}$-discriminant amoeba for $(n+2)$,$(n+3)$ and $(n+4)$ - nomials.

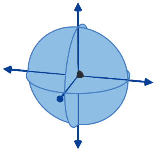
(n + 2) - Case



(n+3) - Case



(n+4) - Case

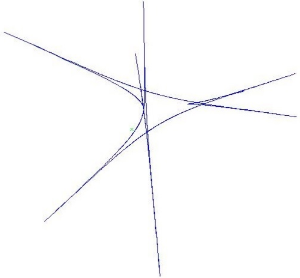


The contour is the image of the real zero set of a polynomial under the $\text{Log}|\cdot|$ map.

Amoeba: Importance

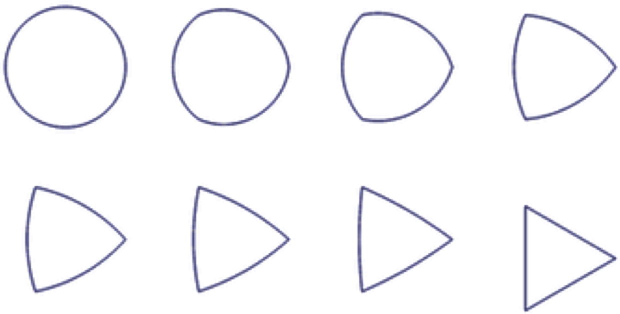
- The complement of the amoeba is the finite disjoint union of open convex sets.
- These unbounded open convex sets are called outer chambers
- The topology of the real zero set is constant in each outer chamber.

Figure: Amoeba($\overline{\Delta}_{\mathcal{A}}$): Polynomial of degree 31, 8 terms



Amoeba: Importance

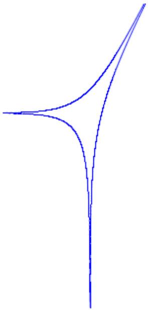
- The topology of the real zero set is constant in each outer chamber.
- The zero sets of the polynomials within each chamber are isotopic.



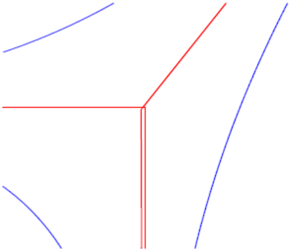
Approximations: Chamber Cones

- Computing an amoeba can be inefficient.
- Instead, we can use an approximation to estimate where the amoeba and it's chambers lie.

Amoeba



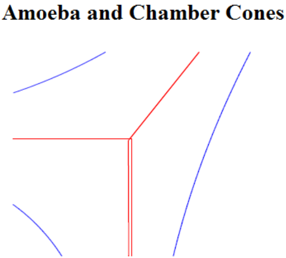
Amoeba and Chamber Cones

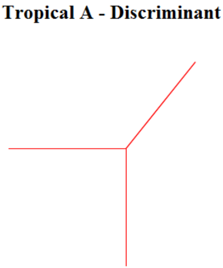


- Chamber cones are used as an approximation of amoeba.

Approximations: Tropical $\mathcal{A}$-Discriminant

- The tropical $\mathcal{A}$-discriminant is the union of cones centered at the origin.





- The tropical $\mathcal{A}$-discriminant can be found more quickly than the chamber cones.

Goals

- Create an algorithm to visualize the reduced $\mathcal{A}$-discriminant amoeba for $(n+4)$ - nomials.

- Create an algorithm to compute the reduced tropical $\mathcal{A}$-discriminant for $(n+4)$ - nomials

Applications

- Polynomial models are used in: robotics, mathematics biology, game theory, statistics and machine learning.
- Certain problems in physical modeling involve solving systems of real polynomial equations.
- Many industrial problems involve sparse polynomial systems whose real roots lie outside the reach of current algorithmic techniques.

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Thank you for listening!