Effective Non-vanishing of Class Group L-Functions for Biquadratic CM Fields

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Notation

Fix the following notation:

- F is a number field.
- d_F is the absolute value of the discriminant of F.
- $\mathcal{O}_F$ is the ring of integers of F.
- $\mathcal{O}_F^\times$ is the group of units of $\mathcal{O}_F$.
- $Cl(\mathcal{O}_F)$ is the ideal class group of F.
- h_F is the class number.
- R_F is the regulator of F.
- $\zeta_F(s)$ is the Dedekind zeta function.
- γ_F is the constant term of $\zeta_F(s)$ at $s = 1$.

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- The *class number* h_F is the order of the class group.
- $\mathcal{O}_F$ is a PID $\iff$ F has class number 1.

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Note. $\#\widehat{G} = \#G$.

Class group L-functions

Definition

Given $\chi \in \widehat{Cl(\mathcal{O}_F)}$, we define the *class group L-function* by

$$L(\chi, s) = \sum_{C \in Cl(\mathcal{O}_F)} \chi(C)\zeta_F(s, C)$$

where

$$\zeta_F(s, C) = \sum_{0 \neq I \in C} N(I)^{-s}$$

is the *partial zeta function.*

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- One has a functional equation relating $L(\chi, s)$ to $L(\chi, 1-s)$.
- The "central value" is $L(\chi, \frac{1}{2})$.
- We wish to determine whether $L(\chi, \frac{1}{2}) \neq 0$.

Eisenstein series

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- Let $\mathbf{y} = \mathrm{Im}(\mathbf{z}) = (y_1, \ldots, y_n)$ and $N(\mathbf{y}) = \prod_{j=1}^{n} y_j$.
- Let $N(\alpha + \beta\mathbf{z}) = \prod_{j=1}^{n}(\sigma_j(\alpha) + \sigma_j(\beta)z_j)$ for $\alpha, \beta \in K$.

Eisenstein series

Definition

The *Hilbert modular Eisenstein series* is defined by

$$E_K(\mathbf{z}, s) = \sum_{0 \neq (\alpha,\beta) \in \mathcal{O}_K^2/\mathcal{O}_K^\times} \frac{N(\mathbf{y})^s}{|N(\alpha + \beta\mathbf{z})|^{2s}}, \quad \mathbf{z} \in \mathbb{H}^n, \quad \operatorname{Re}(s) > 1.$$

The average formula

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Proposition

For $\chi \in Cl(\mathcal{O}_E)$, we have

$$\frac{1}{h_E} \sum_{\chi \in \widehat{Cl(\mathcal{O}_E)}} L(\chi, s) = \left(\frac{2^n d_K}{\sqrt{d_E}}\right)^s \frac{1}{[\mathcal{O}_E^\times : \mathcal{O}_K^\times]} E_K(\mathbf{z}_{\mathcal{O}_E}, s),$$

where $\mathbf{z}_{\mathcal{O}_E} \in \mathbb{H}^n$ is a certain special point depending on $\mathcal{O}_E$.

Statement of Main Result

Theorem (B-S,P,Weber)

Let $d_1 > 0$ and $d_2 < 0$ be squarefree, coprime integers with $d_1 \equiv 1 \mod 4$ and $d_2 \equiv 2$ or $3 \mod 4$. Assume $K = \mathbb{Q}(\sqrt{d_1})$ has class number 1 and let $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Then if

$$|d_2| \geq (318310)^2 d_1 \exp\left\{\sqrt{d_1}(\log(4d_1) + 2\right\}$$

then there exists a character $\chi \in \widehat{Cl(\mathcal{O}_E)}$ such that $L(\chi, \frac{1}{2}) \neq 0$.