Zeros of Maass forms

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Number Theory REU

July 24, 2014

Introduction

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- There have been many interesting results concerning the zeros of various types of modular forms.
- Maass forms are a generalization of modular forms.
- Our aim is to determine the location and number of zeros of a Maass form inside the fundamental domain $\mathcal{F} = \{z \in \mathbb{H} : -1/2 \leq \text{Re}(z) \leq 1/2, |z| \geq 1\}$.

Preliminaries

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For Maass forms, the second condition is replaced by:

2'. f is an eigenfunction of the operator $-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

Eisenstein series

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$$E_k(z) = \frac{1}{2} \sum_{\substack{\gcd(c,d)=1 \\ c,d \in \mathbb{Z}}} \sum (cz+d)^{-k} \tag{1}$$

We can also study its Fourier expansion, given by

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}. \tag{2}$$

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- If f transforms like a weight k modular form, then $R_k f$ transforms like a weight $k+2$ modular form.
- We are interested in the properties of $R_k\left(E_k(z)\right)$. In particular, we want to study the amount and the location of its zeros inside the fundamental domain $\mathcal{F}$.

Maass weight raising operator

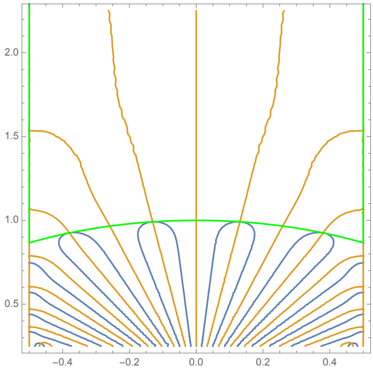
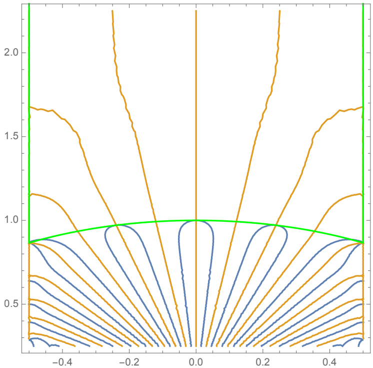


Figure : $\mathrm{Re}(E_{24}(z)) = 0$ *in blue and* $\mathrm{Im}(E_{24}(z)) = 0$ *in yellow.*



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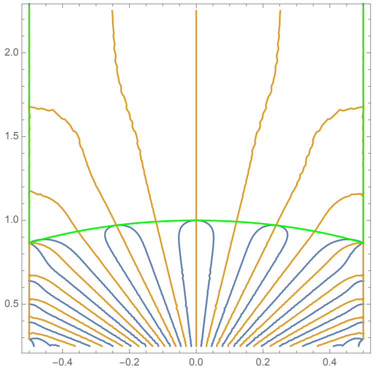
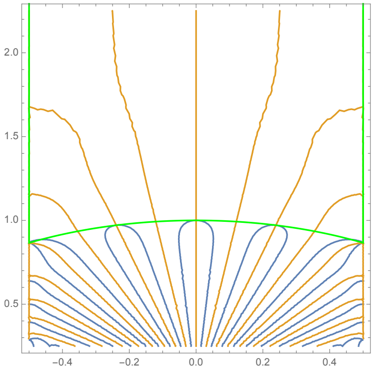


Figure : $\mathrm{Re}(E_{26}(z)) = 0$ *in blue and* $\mathrm{Im}(E_{26}(z)) = 0$ *in yellow.*



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Previous results about zeros of E_k

- If f is a modular form of weight k, the valence formula is given by

$$\frac{k}{12} = \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\rho(f) + \mathrm{ord}_\infty(f) + \sum_{\tau \in \Gamma\backslash\mathbb{H} - \{i,\rho\}} \mathrm{ord}_\tau(f).$$

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- Note that if we write $k = 12m(k) + s$ where $s = 4, 6, 8, 10, 0$ or 14, then s determines the residue class of k modulo 12.

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- Note that if we write $k = 12m(k) + s$ where $s = 4, 6, 8, 10, 0$ or 14, then s determines the residue class of k modulo 12.
- In [1], Rankin and Swinnerton-Dyer proved that all the zeros of $E_k(z)$ in the fundamental domain $\mathcal{F}$ lie on the arc $\mathcal{A} = \left\{e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\right\}$.

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- Note that $R_k E_k$ is no longer a holomorphic function and therefore we the valence formula does not hold.
- Hence, this theorem does not exclude the possibility of $R_k E_k$ having other zeros on $\mathcal{F}$.

Zeros of $R_{68}E_{68}$

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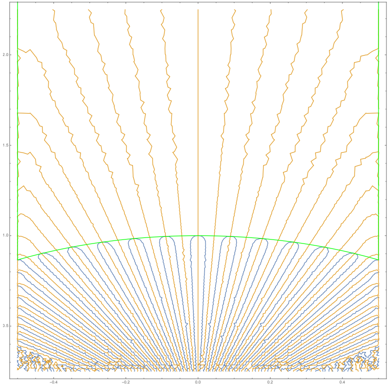

Figure : $\mathrm{Re}(R_{68}E_{24}(z)) = 0$ *in blue and* $\mathrm{Im}(R_{68}E_{68}(z)) = 0$ *in yellow.*

Some conjectures

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Let $R_k^j = R_{k+2j-2} \circ \cdots \circ R_{k+2} \circ R_k$. Then $R_k^j(E_k)$ has the same amount of zeros as E_{k+2j}. Furthermore, all of the zeros of $R_k^j(E_k)$ inside the fundamental domain lie on the arc $\mathcal{A}$.

F. K. C. Rankin and H. P. F. Swinnerton-Dyer.
On the zeros of Eisenstein series.
Bulletin of the London Mathematical Society, 2:169–170, 1970.