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Effective Bounds for Traces of Singular Moduli

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July 16, 2018

DMS-1757872

Thank you

Effective Bounds for Traces of Singular Moduli

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Related Theorems Zagier Duke

A Result Statement of Result Comparison

A Proof of the Result Reduced Forms The Poincaré Series A Useful Proposition Bounding $Tr_d(J)$ We would like thank Riad Masri for his guidance and advice while conducting this research. We would also like to thank Texas A&M's Department of Mathematics for their hospitality during this summer of research. And lastly we would like to thank the NSF for supporting us in this incredible opportunity to learn and directly interact with beautiful math.

Outline

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 \bullet Let $\mathbb H$ denote the complex upper half plane.

The upper half plane and modular group

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A Proof of the Result Reduced Forms The Poincaré Series A Useful • Let \mathbb{H} denote the complex upper half plane.

$$ullet$$
 Let $\mathsf{SL}_2(\mathbb{Z}) = \left\{ egin{pmatrix} a & b \ c & d \end{pmatrix} ig| a, b, c, d \in \mathbb{Z}, ad-bc = 1
ight\}$

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The Poincaré Series A Useful Proposition Bounding $Tr_d(J)$ • Let \mathbb{H} denote the complex upper half plane.

$$ullet$$
 Let $\mathsf{SL}_2(\mathbb{Z}) = \left\{ egin{pmatrix} a & b \ c & d \end{pmatrix} ig| a, b, c, d \in \mathbb{Z}, ad-bc=1
ight\}$

• $SL_2(\mathbb{Z})$ acts on \mathbb{H} by linear fractional transformations: If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$, then the group action is defined by

$$\gamma(z) = \frac{az+b}{cz+d}.$$

The J-function

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A Proof of the Result Reduced Forms The Poincaré Series A Useful Proposition • The classical modular *j*-function is defined as

$$j(z) := e(-z) + 744 + \sum_{n>0} a(n)e(nz), \quad z \in \mathbb{H}$$

where $e(z) := e^{2\pi i z}$ and $a(n) \in \mathbb{Z}$ is a Fourier coefficient for which an explicit formula can be found.

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A Proof of the Result Reduced Forms The Poincaré Series A Useful Proposition • The classical modular *j*-function is defined as

$$j(z) := e(-z) + 744 + \sum_{n>0} a(n)e(nz), \quad z \in \mathbb{H}$$

where $e(z):=e^{2\pi iz}$ and $a(n)\in\mathbb{Z}$ is a Fourier coefficient for which an explicit formula can be found.

• Define J(z) := j(z) - 744.

The J-function

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• The classical modular *j*-function is defined as

$$j(z) := e(-z) + 744 + \sum_{n>0} a(n)e(nz), \quad z \in \mathbb{H}$$

where $e(z) := e^{2\pi i z}$ and $a(n) \in \mathbb{Z}$ is a Fourier coefficient for which an explicit formula can be found.

- Define J(z) := j(z) 744.
- Note that

$$J(\gamma z)=J(z)$$

for all $\gamma \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$, and so J is an automorphic function.

Binary Quadratic Forms

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A Proof of the Result Reduced Forms The Poincaré Series A Useful Proposition • Let Q_d be the set of primitive, positive-definite, integral, binary quadratic forms

$$Q(x,y) = [a_Q,b_Q,c_Q] = a_Q x^2 + b_Q xy + c_Q y^2$$
 with discriminant $d=b_Q^2 - 4a_Q c_Q < 0$.

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• Let Q_d be the set of primitive, positive-definite, integral, binary quadratic forms

$$Q(x,y) = [a_Q, b_Q, c_Q] = a_Q x^2 + b_Q xy + c_Q y^2$$

with discriminant $d = b_Q^2 - 4a_Qc_Q < 0$.

ullet There is a right action of $\mathsf{SL}_2(\mathbb{Z})$ on Q_d given by

$$Q \circ M(x, y) = Q(\alpha x + \beta y, \gamma x + \delta y)$$

where
$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}).$$

The Class Number

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The Poincaré Series A Useful Proposition • The quotient $Q_d/\operatorname{SL}_2(\mathbb{Z})$ is finite. Let

$$h(d) := |Q_d/\mathsf{SL}_2(\mathbb{Z})|$$

be the *class number* of *d*.

The Class Number

Effective Bounds for Traces of Singular Moduli

Definitions

• The quotient $Q_d/\operatorname{SL}_2(\mathbb{Z})$ is finite. Let

$$h(d) := |Q_d/\mathsf{SL}_2(\mathbb{Z})|$$

be the class number of d.

Theorem (Siegel)

For all $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that

$$h(d) \geq C(\epsilon) |d|^{\frac{1}{2}+\epsilon}$$
.

Singular Moduli

Effective Bounds for Traces of Singular Moduli

Definitions

• We are interested in evaluating the *J*-function at certain distinguished algebraic integers.

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- We are interested in evaluating the *J*-function at certain distinguished algebraic integers.
- A *CM point* is the root of Q(x,1) in \mathbb{H} given by

$$\tau_Q = \frac{-b_Q + i\sqrt{|d|}}{2a_Q}.$$

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Bounding $\pi_{r_{\ell}}(J)$

- We are interested in evaluating the *J*-function at certain distinguished algebraic integers.
- A *CM* point is the root of Q(x,1) in \mathbb{H} given by

$$\tau_Q = \frac{-b_Q + i\sqrt{|d|}}{2a_Q}.$$

• The values $J(\tau_Q)$ are algebraic numbers called *singular moduli*.

Traces of singular moduli

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• We define the trace of singular moduli by:

$$\mathit{Tr}_d(J) := \sum_{[Q] \in Q_d/\operatorname{SL}_2(\mathbb{Z})} J(\tau_Q).$$

Traces of singular moduli

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Bounding $Tr_d(J)$

We define the trace of singular moduli by:

$$Tr_d(J) := \sum_{[Q] \in Q_d/\mathrm{SL}_2(\mathbb{Z})} J(\tau_Q).$$

• The trace is well defined because if $[Q_1] = [Q_2]$, then $\gamma \tau_{Q_1} = \tau_{Q_2}$ for some $\gamma \in \mathsf{SL}_2(\mathbb{Z})$, and J is automorphic.

Zagier's generating function

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The Poincaré Series A Useful Proposition Let

$$g_{Zag}(z) := e(-z|d|) + \sum_{d \equiv 0,1 \pmod{4}} Tr_d(J)e(z|d|).$$

Zagier's generating function

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Let

$$g_{Zag}(z) := e(-z|d|) + \sum_{d \equiv 0,1 \pmod{4}} Tr_d(J)e(z|d|).$$

• A remarkable theorem of Zagier asserts that $g_{Zag}(z)$ is a weakly holomorphic modular form of weight 3/2 for $\Gamma_0(4)$.

Importance

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 Zagier's theorem tells us that using traces of singular moduli, we can construct a new weakly holomorphic modular form of a different weight.

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- Zagier's theorem tells us that using traces of singular moduli, we can construct a new weakly holomorphic modular form of a different weight.
- A problem of central importance in number theory is to bound Fourier coefficients of modular forms.

Importance

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Bounding Tra(J)

- Zagier's theorem tells us that using traces of singular moduli, we can construct a new weakly holomorphic modular form of a different weight.
- A problem of central importance in number theory is to bound Fourier coefficients of modular forms.
- As a consequence of our main theorem, we will give effective bounds for the Fourier coefficients of $g_{Zag}(z)$.

A Theorem of Duke

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Theorem (Duke, 2006)

There is an absolute constant $\delta > 0$ such that

$$Tr_d(J) = \sum_{\substack{[Q] \in Q_d/SL_2(\mathbb{Z}) \ \operatorname{Im}(au_Q) > 1}} e(- au_Q) - 24h(d) + \mathcal{O}(|d|^{rac{1}{2}-\delta}).$$

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• Note that $\frac{\mathcal{O}(|d|^{\frac{1}{2}-\delta})}{h(d)} \to 0$ as $|d| \to \infty$ by Siegel's Theorem.

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Proposition

• Note that $\frac{\mathcal{O}(|d|^{\frac{1}{2}-\delta})}{h(d)} \to 0$ as $|d| \to \infty$ by Siegel's Theorem.

Thus Duke's theorem implies that

$$\frac{Tr_d(J) - \sum_{\substack{[Q] \in Q_d / \mathsf{SL}_2(\mathbb{Z}) \\ \operatorname{Im}(\tau_Q) > 1}} e(-\tau_Q)}{h(d)} \to -24$$

as $|d| \to \infty$. This confirmed a conjecture of Bruinier, Jenkins, and Ono.

Special case of our Main Theorem

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Theorem

$$Tr_d(J) = \sum_{\substack{[Q] \in Q_d/SL_2(\mathbb{Z}) \ \operatorname{Im}(\tau_Q) > 1}} e(-\tau_Q) - 24h(d) + E(d)$$

where

$$|E(d)| \leq (1.72 \times 10^6) h(d).$$

A corollary

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A Useful Proposition Bounding $Tr_d(J)$

Corollary

$$|\mathit{Tr}_d(J)| \le e^{\pi \sqrt{|d|}} (1.72 \times 10^6) h(d)$$

Comparison with Duke's Theorem

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Duke proved that

$$Tr_d(J) - \sum_{\substack{[Q] \in Q_d / \operatorname{SL}_2(\mathbb{Z}) \\ \operatorname{Im}(au_Q) > 1}} e(- au_Q)$$

converges by saving a power of d in the error term over the "trivial" bound $h(d) \ll \log(|d|)\sqrt{|d|}$.

Comparison with Duke's Theorem

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$$Tr_d(J) - \sum_{\substack{[Q] \in Q_d/\operatorname{SL}_2(\mathbb{Z}) \\ \operatorname{Im}(\tau_Q) > 1}} e(-\tau_Q)$$

converges by saving a power of d in the error term over the "trivial" bound $h(d) \ll \log(|d|) \sqrt{|d|}$.

 However because of the methods involved in Duke's proof, one cannot practically compute the implied constant in his error term.

Comparison with Duke's Theorem

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Duke proved that

$$Tr_d(J) - \sum_{\substack{[Q] \in Q_d/\operatorname{SL}_2(\mathbb{Z}) \\ \operatorname{Im}(au_Q) > 1}} e(- au_Q)$$

converges by saving a power of d in the error term over the "trivial" bound $h(d) \ll \log(|d|) \sqrt{|d|}$.

- However because of the methods involved in Duke's proof, one cannot practically compute the implied constant in his error term.
- Therefore we require a new method for our main theorem.

Reduced forms

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• The fundamental domain for $SL_2(\mathbb{Z})$ acting on \mathbb{H} is the region

$$\mathcal{F} := \left\{ z \in \mathbb{C} \left| \ |z| > 1 \text{ and } -\frac{1}{2} \le \operatorname{Re}(z) < \frac{1}{2} \right\}$$

$$\cup \left\{ z \in \mathbb{C} \left| \ -\frac{1}{2} \le \operatorname{Re}(z) \le 0, |z| = 1 \right\}. \right.$$

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 \bullet The fundamental domain for $\mathsf{SL}_2(\mathbb{Z})$ acting on \mathbb{H} is the region

$$\mathcal{F} := \left\{ z \in \mathbb{C} \left| \ |z| > 1 \text{ and } -\frac{1}{2} \le \operatorname{Re}(z) < \frac{1}{2} \right\}$$

$$\cup \left\{ z \in \mathbb{C} \left| \ -\frac{1}{2} \le \operatorname{Re}(z) \le 0, |z| = 1 \right\}.$$

• A form Q is said to be *reduced* if its CM point lies in \mathcal{F} .

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 \bullet The fundamental domain for $\mathsf{SL}_2(\mathbb{Z})$ acting on \mathbb{H} is the region

$$\mathcal{F} := \left\{ z \in \mathbb{C} \left| \; |z| > 1 \; \mathsf{and} \; - rac{1}{2} \leq \mathrm{Re}(z) < rac{1}{2}
ight\} \ \ \cup \left\{ z \in \mathbb{C} \left| \; - rac{1}{2} \leq \mathrm{Re}(z) \leq 0, |z| = 1
ight\}.$$

- ullet A form Q is said to be *reduced* if its CM point lies in \mathcal{F} .
- Each $[Q] \in Q_d/\mathsf{SL}_2(\mathbb{Z})$ contains a unique reduced form.

Summing over reduced forms

Effective Bounds for Traces of Singular Moduli

Reduced Forms

• Let $Q_1, \ldots, Q_{h(d)}$ be the set of reduced forms representing the equivalence classes in $Q_d/SL_2(\mathbb{Z})$.

Summing over reduced forms

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- Let $Q_1, \ldots, Q_{h(d)}$ be the set of reduced forms representing the equivalence classes in $Q_d/\mathrm{SL}_2(\mathbb{Z})$.
- We can sum over $Q_1, \ldots, Q_{h(d)}$ in the trace of J(z):

$$Tr_d(J) = \sum_{i=1}^{h(d)} J(\tau_{Q_i}).$$

The Poincaré series

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Delilitioi

Related Theorem

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Statement of Result Comparison

A Proof of the Result Reduced Forms The Poincaré Series A Useful Proposition • For $s \in \mathbb{C}$ with Re(s) > 1 and $z \in \mathbb{H}$, define the Maass-Poincaré series

$$F(z,s) := 2\pi \sum_{\gamma \in \Gamma_{\infty} \setminus \mathsf{SL}_2(\mathbb{Z})} \mathsf{Im}(\gamma z)^{rac{1}{2}} I_{s-rac{1}{2}}(2\pi \mathsf{Im}(\gamma z)) \mathsf{e}(-\mathsf{Re}(\gamma z))$$

The Poincaré series

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The Poincaré

• For $s \in \mathbb{C}$ with Re(s) > 1 and $z \in \mathbb{H}$, define the Maass-Poincaré series

$$F(z,s) := 2\pi \sum_{\gamma \in \Gamma_{\infty} \setminus \mathsf{SL}_2(\mathbb{Z})} \mathsf{Im}(\gamma z)^{rac{1}{2}} I_{s-rac{1}{2}}(2\pi \mathsf{Im}(\gamma z)) \mathsf{e}(-\mathsf{Re}(\gamma z))$$

• I_{ν} is the I Bessel function of order ν .

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• For $s \in \mathbb{C}$ with Re(s) > 1 and $z \in \mathbb{H}$, define the Maass-Poincaré series

$$F(z,s) := 2\pi \sum_{\gamma \in \Gamma_{\infty} \setminus \mathsf{SL}_2(\mathbb{Z})} \mathsf{Im}(\gamma z)^{rac{1}{2}} I_{s-rac{1}{2}}(2\pi \mathsf{Im}(\gamma z)) e(-\mathsf{Re}(\gamma z))$$

- I_{ν} is the I Bessel function of order ν .
- And

$$\Gamma_{\infty} := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z}^+ \cup \{0\} \right\}$$

is the subset of $SL_2(\mathbb{Z})$ that stabilizes the cusp at infinity.

Proposition

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Proposition

The limit

$$\lim_{s\to 1^+} F(z,s)$$

exists and is given by

$$F(z,1) = e(-z) + \sum_{n=0}^{\infty} b(n)e(nz)$$

where b(0) = 24 and

$$b(n) = 2\pi n^{-\frac{1}{2}} \sum_{r=0}^{\infty} \frac{S(n,-1;c)}{c} I_1\left(\frac{4\pi\sqrt{n}}{c}\right), \qquad n > 0.$$

• J(z) = F(z,1) - 24.

The Kloosterman sum

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A Useful Proposition Bounding $Tr_a(J)$ • S(a, b; c) is the ordinary Kloosterman sum

$$S(a, b; c) := \sum_{\substack{d \pmod{c} \\ (c,d)=1}} e\left(\frac{a\overline{d} + bd}{c}\right)$$

where \overline{d} is the multiplicative inverse of $d \pmod{c}$.

The Fourier expansion

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A Useful Proposition Bounding $Tr_d(J)$ F(z, s) has a Fourier expansion given by

$$F(z,s) = 2\pi y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y)e(-x) + c_s y^{1-s} + 4\pi \sum_{n\neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y)e(nx)$$

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A Useful Proposition Bounding $Tr_a(J)$ F(z, s) has a Fourier expansion given by

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where

$$c_s := \frac{4\pi^{1+s}}{(2s-1)\Gamma(s)\zeta(2s)}$$

The Fourier expansion

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A Useful Proposition Bounding $Tr_d(J)$ F(z, s) has a Fourier expansion given by

$$F(z,s) = 2\pi y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y)e(-x) + c_s y^{1-s} + 4\pi \sum_{n \neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y)e(nx)$$

where

$$c_s := \frac{4\pi^{1+s}}{(2s-1)\Gamma(s)\zeta(2s)}$$

and

$$b(n;s) := \sum_{c>0} \frac{S(n,-1;c)}{c} \begin{cases} I_{2s-1}\left(\frac{4\pi\sqrt{n}}{c}\right) & n>0\\ J_{2s-1}\left(\frac{4\pi\sqrt{|n|}}{c}\right) & n<0. \end{cases}$$

The first two terms

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A Useful Proposition Bounding $Tr_d(J)$ $F(z,s) = 2\pi y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y) e(-x) + c_s y^{1-s}$ $+ 4\pi \sum_{n \neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx)$

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A Useful Proposition Bounding $Tr_a(J)$

$$F(z,s) = 2\pi y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y) e(-x) + c_s y^{1-s}$$

$$+ 4\pi \sum_{n \neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx)$$

• These are analytic functions on C.

The first two terms

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A Useful Proposition Bounding $Tr_d(x)$ $F(z,s) = 2\pi y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y) e(-x) + c_s y^{1-s}$ $+ 4\pi \sum_{n \neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx)$

- These are analytic functions on C.
- We want to show that for $z \in \mathbb{H}$, the sum

$$B(z,s) := \sum_{n \neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx)$$

converges absolutely for all $s \in \mathbb{R}$ such that $s \ge 1$.

Bounding the Fourier coefficients

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A Useful Proposition **Proposition**

For $s \in \mathbb{R}$ such that $s \geq 1$,

$$|b(n;s)| \le \begin{cases} C_1(s) |n|^s & n < 0 \\ C_2(s) n^s e^{4\pi\sqrt{n}} & n > 0 \end{cases}$$

and

$$\left| K_{s-\frac{1}{2}}(2\pi |n| y) \right| \le C_3(s) \frac{e^{-2\pi |n| y}}{\sqrt{|n| y}}$$

where C_1 , C_2 , and C_3 are explicit constants that depend on s.

Key ideas

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A Useful Proposition Bounding $Tr_d(J)$ The Weil bound:

$$|S(a,b;c)| \le \tau(c)(a,b,c)^{1/2}c^{1/2}$$

where au is the divisor function.

Key ideas

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A Useful Proposition Bounding $Tr_a(J)$ • The Weil bound:

$$|S(a,b;c)| \le \tau(c)(a,b,c)^{1/2}c^{1/2}$$

where au is the divisor function.

 A careful study of the asymptotics of the I, J, and K Bessel functions.

Bounding the infinite sum

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A Useful Proposition Bounding $Tr_a(J)$ Using these bounds we can show that for $z \in \mathbb{H}$,

$$|B(z,s)| \le \sum_{n \ne 0} \left| b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}} (2\pi |n| y) e(nx) \right| < \infty$$

for all $s \in \mathbb{R}$ such that $s \geq 1$.

The Fourier expansion of F(z, 1)

Effective Bounds for Traces of Singular Moduli

A Useful Proposition Thus after some manipulation we find that

$$\lim_{s \to 1^{+}} F(z, s) = F(z, 1) = e(-z) + 24 - e(-\overline{z})$$

$$+ 2\pi \sum_{n < 0} b(n; 1) |n|^{-\frac{1}{2}} e(n\overline{z})$$

$$+ 2\pi \sum_{n > 0} b(n; 1) n^{-\frac{1}{2}} e(nz).$$

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A Useful

Let

$$\phi(z) := F(z,1) - J(z).$$

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A Useful Proposition Let

$$\phi(z) := F(z,1) - J(z).$$

Recall:

$$J(z) := e(-z) + \sum_{n>0} a(n)e(nz).$$

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A Useful Proposition Bounding $Tr_a(J)$ Let

$$\phi(z) := F(z,1) - J(z).$$

Recall:

$$J(z) := e(-z) + \sum_{n>0} a(n)e(nz).$$

• Note that F(z,1) and J(z) have the same principal part.

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Proposition

Let

$$\phi(z) := F(z,1) - J(z).$$

Recall:

$$J(z) := e(-z) + \sum_{n>0} a(n)e(nz).$$

- Note that F(z,1) and J(z) have the same principal part.
- Hence the function $\phi(z)$ is bounded on \mathbb{H} .

The hyperbolic Laplacian operator

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A Useful Proposition Bounding $T_{r_d}(J)$ • The hyperbolic Laplacian is

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The hyperbolic Laplacian operator

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A Proof of th Result Reduced Forms The Poincaré Series A Useful Proposition Bounding Tra(J • The hyperbolic Laplacian is

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

• Fact: If f is a holomorphic function on \mathbb{H} then $\Delta f(z) = 0$.

The hyperbolic Laplacian operator

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$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

- Fact: If f is a holomorphic function on \mathbb{H} then $\Delta f(z) = 0$.
- Since J(z) is holomorphic on \mathbb{H} , $\Delta J(z) = 0$.

$\phi(z)$ is harmonic

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A Useful Proposition Bounding Tr_d(J) It is known that

$$\Delta F(z,s) = s(s-1)F(z,s).$$

$\phi(z)$ is harmonic

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A Useful Proposition Bounding $Tr_d(J)$ It is known that

$$\Delta F(z,s) = s(s-1)F(z,s).$$

• So $\Delta F(z,1) = 0$.

$\phi(z)$ is harmonic

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A Useful Proposition Bounding $Tr_d(J)$ It is known that

$$\Delta F(z,s) = s(s-1)F(z,s).$$

- So $\Delta F(z,1) = 0$.
- Therefore $\Delta \phi(z) = 0$, so $\phi(z)$ is harmonic.

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A Useful Proposition Bounding $Tr_d(J)$ • Fact: A bounded harmonic function on \mathbb{H} is constant.

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A Useful Proposition Bounding $Tr_d(J)$ • Fact: A bounded harmonic function on \mathbb{H} is constant.

So $\phi(z) = C$ for some constant C.

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A Useful Proposition Bounding $Tr_a(J)$ • Fact: A bounded harmonic function on \mathbb{H} is constant.

So $\phi(z) = C$ for some constant C.

Since

$$CT(J(z)) = 0$$
 and $CT(F(z,1)) = 24$

we have that

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Proposition

• Fact: A bounded harmonic function on \mathbb{H} is constant.

So $\phi(z) = C$ for some constant C.

Since

$$CT(J(z)) = 0$$
 and $CT(F(z,1)) = 24$

we have that

$$\phi(z) = F(z,1) - J(z) = 24$$

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A Useful Proposition Bounding $Tr_d(J)$ • Fact: A bounded harmonic function on \mathbb{H} is constant.

So $\phi(z) = C$ for some constant C.

Since

$$CT(J(z)) = 0$$
 and $CT(F(z,1)) = 24$

we have that

$$\phi(z) = F(z,1) - J(z) = 24$$

and thus

$$J(z)=F(z,1)-24.$$

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Bounding Tr_d(J)

ullet Fact: A bounded harmonic function on $\mathbb H$ is constant.

So $\phi(z) = C$ for some constant C.

Since

$$CT(J(z)) = 0$$
 and $CT(F(z,1)) = 24$

we have that

$$\phi(z) = F(z,1) - J(z) = 24$$

and thus

$$J(z) = F(z,1) - 24.$$

• This proves the second part of the proposition.

The anti-holomorphic part

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A Useful Proposition Bounding $Tr_d(J)$ Recall:

$$F(z,1) = e(-z) + 24 - e(-\overline{z})$$

$$+2\pi \sum_{n<0} b(n;1) |n|^{-\frac{1}{2}} e(n\overline{z})$$

$$+2\pi \sum_{n>0} b(n;1) n^{-\frac{1}{2}} e(nz).$$

The anti-holomorphic part (cont.)

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Proposition

• Since F(z,1) - 24 = J(z) and J(z) is holomorphic, the anti-holomorphic part of F(z,1) is zero, hence

$$F(z,1) = e(-z) + 24 + 2\pi \sum_{n>0} b(n;1)n^{-\frac{1}{2}}e(nz)$$

The anti-holomorphic part (cont.)

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A Useful Proposition Bounding $Tr_a(J)$ • Since F(z,1) - 24 = J(z) and J(z) is holomorphic, the anti-holomorphic part of F(z,1) is zero, hence

$$F(z,1) = e(-z) + 24 + 2\pi \sum_{n>0} b(n;1)n^{-\frac{1}{2}}e(nz)$$

• We can conclude that b(0) = 24 and

$$b(n) = 2\pi b(n; 1)n^{-\frac{1}{2}}$$

$$= 2\pi n^{-\frac{1}{2}} \sum_{c>0} \frac{S(n, -1; c)}{c} I_1\left(\frac{4\pi\sqrt{n}}{c}\right), \qquad n > 0$$

Bounding the trace of J(z)

Effective Bounds for Traces of Singular Moduli

Bounding $Tr_d(J)$

The trace of J(z) is

$$Tr_d(J(z)) = \sum_{i=1}^{h(d)} (F(\tau_{Q_i}, 1) - 24)$$

Bounding the trace of J(z)

Effective Bounds for Traces of Singular Moduli

Bounding $Tr_d(J)$

The trace of J(z) is

$$Tr_d(J(z)) = \sum_{i=1}^{h(d)} (F(\tau_{Q_i}, 1) - 24)$$

= $Tr_d(F(z, 1)) - 24h(d)$

Bounding the trace of J(z)

Effective Bounds for Traces of Singular Moduli

Bounding $Tr_d(J)$

The trace of J(z) is

$$Tr_{d}(J(z)) = \sum_{i=1}^{h(d)} (F(\tau_{Q_{i}}, 1) - 24)$$

$$= Tr_{d}(F(z, 1)) - 24h(d)$$

$$= \sum_{i=1}^{h(d)} e(-\tau_{Q_{i}}) - 24h(d) + E(d)$$

Bounding the trace of J(z)

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A Useful Proposition Bounding Tr_d(J) The trace of J(z) is

$$Tr_{d}(J(z)) = \sum_{i=1}^{h(d)} (F(\tau_{Q_{i}}, 1) - 24)$$

$$= Tr_{d}(F(z, 1)) - 24h(d)$$

$$= \sum_{i=1}^{h(d)} e(-\tau_{Q_{i}}) - 24h(d) + E(d)$$

where

$$E(d) := \sum_{n=0}^{\infty} b(n) \sum_{i=1}^{h(d)} e(n\tau_{Q_i}).$$

Effective Bounds for Traces of Singular Moduli

Bounding $Tr_d(J)$

We can write

$$\sum_{i=1}^{h(d)} e(- au_{Q_i}) = \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) > 1}} e(- au_{Q_i}) + \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} e(- au_{Q_i}).$$

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We can write

$$\sum_{i=1}^{h(d)} e(- au_{Q_i}) = \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) > 1}} e(- au_{Q_i}) + \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} e(- au_{Q_i}).$$

Note that

$$\bigg|\sum_{\substack{Q_i\\ \operatorname{Im}(\tau_{Q_i}) \leq 1}} e(-\tau_{Q_i})\bigg| \leq \sum_{\substack{Q_i\\ \operatorname{Im}(\tau_{Q_i}) \leq 1}} |e(-\tau_{Q_i})|$$

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$$\sum_{i=1}^{h(d)} e(- au_{Q_i}) = \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) > 1}} e(- au_{Q_i}) + \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} e(- au_{Q_i}).$$

Note that

$$egin{aligned} \left|\sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} e(- au_{Q_i})
ight| &\leq \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} |e(- au_{Q_i})| \ &= \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} \left|e^{-2\pi i \operatorname{Re}(au_{Q_i})} e^{2\pi \operatorname{Im}(au_{Q_i})}
ight| \end{aligned}$$

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A Useful Proposition Bounding $Tr_d(J)$ We can write

$$\sum_{i=1}^{h(d)} e(- au_{Q_i}) = \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) > 1}} e(- au_{Q_i}) + \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} e(- au_{Q_i}).$$

Note that

$$egin{aligned} \left| \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} e(- au_{Q_i})
ight| & \leq \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} |e(- au_{Q_i})| \ & = \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} \left| e^{-2\pi i \operatorname{Re}(au_{Q_i})} e^{2\pi \operatorname{Im}(au_{Q_i})}
ight| \ & = \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i})}} e^{2\pi \operatorname{Im}(au_{Q_i})} \end{aligned}$$

 $\operatorname{Im}(\tau_{Q_i}) \leq 1$

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Bounding $Tr_d(J)$

We can write

$$\sum_{i=1}^{h(d)} e(- au_{Q_i}) = \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) > 1}} e(- au_{Q_i}) + \sum_{\substack{Q_i \ \operatorname{Im}(au_{Q_i}) \leq 1}} e(- au_{Q_i}).$$

Note that

$$\left| \sum_{\substack{Q_i \\ \operatorname{Im}(\tau_{Q_i}) \leq 1}} e(-\tau_{Q_i}) \right| \leq \sum_{\substack{Q_i \\ \operatorname{Im}(\tau_{Q_i}) \leq 1}} |e(-\tau_{Q_i})|$$

$$= \sum_{\substack{Q_i \\ \operatorname{Im}(\tau_{Q_i}) \leq 1}} \left| e^{-2\pi i \operatorname{Re}(\tau_{Q_i})} e^{2\pi \operatorname{Im}(\tau_{Q_i})} \right|$$

$$= \sum_{\substack{Q_i \\ \operatorname{Im}(\tau_{Q_i}) \leq 1}} e^{2\pi \operatorname{Im}(\tau_{Q_i})} \leq h(d) e^{2\pi}.$$

 $\operatorname{Im}(\tau_{Q_i}) \leq 1$

Bounding |E(d)|

Effective Bounds for Traces of Singular Moduli

Bounding $Tr_d(J)$

First,

$$|E(d)| \leq \sum_{n=0}^{\infty} |b(n)| \sum_{i=1}^{h(d)} |e(n\tau_{Q_i})|.$$

Bounding |E(d)|

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Bounding $Tr_d(J)$

First,

$$|E(d)| \leq \sum_{n=0}^{\infty} |b(n)| \sum_{i=1}^{h(d)} |e(n\tau_{Q_i})|.$$

Now,

$$\sum_{i=1}^{h(d)} |e(n\tau_{Q_i})| = \sum_{i=1}^{h(d)} \left| e^{2\pi i n \operatorname{Re}(\tau_{Q_i})} e^{-2\pi n \operatorname{Im}(\tau_{Q_i})} \right|$$

Bounding |E(d)|

Effective Bounds for Traces of Singular Moduli

Bounding $Tr_d(J)$

First.

$$|E(d)| \leq \sum_{n=0}^{\infty} |b(n)| \sum_{i=1}^{h(d)} |e(n\tau_{Q_i})|.$$

Now.

$$egin{aligned} \sum_{i=1}^{h(d)} |e(n au_{Q_i})| &= \sum_{i=1}^{h(d)} \left| e^{2\pi\mathrm{i}n\mathrm{Re}(au_{Q_i})} e^{-2\pi n\mathrm{Im}(au_{Q_i})}
ight| \ &= \sum_{i=1}^{h(d)} e^{-2\pi n\mathrm{Im}(au_{Q_i})}. \end{aligned}$$

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A Proof of the Result Reduced Forms The Poincaré Series A Useful Proposition Bounding Tra(J) ullet Since $au_{Q_1},\ldots, au_{Q_{h(d)}}$ lie in the fundamental domain ${\mathcal F}$,

$$\operatorname{Im}(\tau_{Q_i}) \geq \frac{\sqrt{3}}{2}$$

for all $1 \le i \le h(d)$,

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ullet Since $au_{Q_1},\ldots, au_{Q_{h(d)}}$ lie in the fundamental domain ${\mathcal F}$,

$$\operatorname{Im}(\tau_{Q_i}) \geq \frac{\sqrt{3}}{2}$$

for all $1 \le i \le h(d)$, and so

$$e^{-2\pi n \operatorname{Im}(\tau_{Q_i})} \leq e^{-\pi n \sqrt{3}}.$$

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ullet Since $au_{Q_1},\ldots, au_{Q_{h(d)}}$ lie in the fundamental domain ${\mathcal F}$,

$$\operatorname{Im}(\tau_{Q_i}) \geq \frac{\sqrt{3}}{2}$$

for all $1 \le i \le h(d)$, and so

$$e^{-2\pi n \operatorname{Im}(\tau_{Q_i})} \leq e^{-\pi n \sqrt{3}}.$$

Thus

$$\sum_{i=1}^{h(d)} e^{-2\pi n \text{Im}(\tau_{Q_i})} \le h(d) e^{-\pi n \sqrt{3}}.$$
 (1)

Effective Bounds for Traces of Singular Moduli

Bounding $Tr_d(J)$

• **Recall:** For $s \in \mathbb{R}$ such that s > 1,

$$|b(n;s)| \le \begin{cases} C_1(s) |n|^s & n < 0 \\ C_2(s) n^s e^{4\pi\sqrt{n}} & n > 0. \end{cases}$$

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Proposition
Bounding Tr_d(J)

• **Recall:** For $s \in \mathbb{R}$ such that $s \ge 1$,

$$|b(n;s)| \le \begin{cases} C_1(s) |n|^s & n < 0 \\ C_2(s) n^s e^{4\pi\sqrt{n}} & n > 0. \end{cases}$$

• So, setting s = 1,

$$|b(n;1)| \le (105.20)ne^{4\pi\sqrt{n}}, \qquad n > 0.$$
 (2)

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Bounding $Tr_d(J)$

• Combining (1) and (2), we get

$$|E(d)| \leq \sum_{n=0}^{\infty} |b(n)| \sum_{i=1}^{h(d)} |e(n\tau_{Q_i})| \leq (1.72 \times 10^6) h(d).$$

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The Poincaré Series A Useful Proposition Bounding Tr_d(J) • Combining (1) and (2), we get

$$|E(d)| \leq \sum_{n=0}^{\infty} |b(n)| \sum_{i=1}^{h(d)} |e(n\tau_{Q_i})| \leq (1.72 \times 10^6) h(d).$$

Combined with our earlier observation that

$$\sum_{i=1}^{h(d)} e(-\tau_{Q_i}) = \sum_{\substack{Q_i \\ \text{Im}(\tau_{Q_i}) > 1}} e(-\tau_{Q_i}) + \sum_{\substack{Q_i \\ \text{Im}(\tau_{Q_i}) \leq 1}} e(-\tau_{Q_i})$$

this completes the proof of the theorem.

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The Poincaré Series A Useful Proposition Bounding Tra(J)

Theorem

$$Tr_d(J) = \sum_{\substack{[Q] \in Q_d/SL_2(\mathbb{Z}) \\ \operatorname{Im}(\tau_Q) > 1}} e(-\tau_Q) - 24h(d) + E(d)$$

where

$$|E(d)| \leq (1.72 \times 10^6) h(d).$$