Patterns arising in the Kernel of Generalized Dedekind Sums (Part 2)

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Dedekind Sum

Congruence Subgroups

$$\Gamma_0(q_1q_2) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 (\bmod q_1q_2)\right\}$$

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Let χ_1, χ_2 be non-trivial primitive Dirichlet characters modulo q_1 and q_2, respectively, such that $q_1, q_2 > 1$ and $\chi_1\chi_2(-1) = -1$. The **generalized Dedekind sum** is

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_1,\chi_2}(a,c) = \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j)\, \overline{\chi_1}(n)\, B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)$$

where $\gamma \in \Gamma_0(q_1q_2)$.

Known points in Kernel: $(1, N)$

Proposition

$S_{\chi_1,\chi_2}(1, c'q_1q_2) = 0$ for all $c' \in \mathbb{Z}$.

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Proof.

For $\gamma = \begin{pmatrix} 1 & 0 \\ c'q_1q_2 & 1 \end{pmatrix}$, let $\gamma' = \begin{pmatrix} 1 & -c' \\ 0 & 1 \end{pmatrix}$. Then if χ_1, χ_2 are even, then

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_2,\chi_1}(\gamma').$$

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If χ_1, χ_2 are odd, then

$$S_{\chi_1,\chi_2}(\gamma) = -S_{\chi_2,\chi_1}(\gamma') + (1 - \psi(\gamma))\left(\frac{\tau(\overline{\chi}_1)\tau(\overline{\chi}_2)}{(\pi i)^2}\right)L(1, \chi_1)L(1, \chi_2).$$

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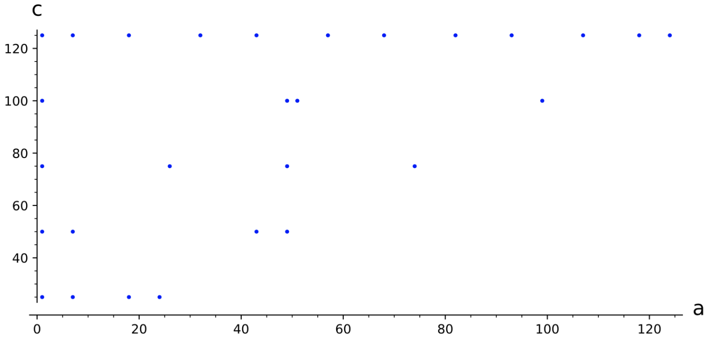

Figure: $\chi_1 \bmod 5$ & $\chi_3 \bmod 11$

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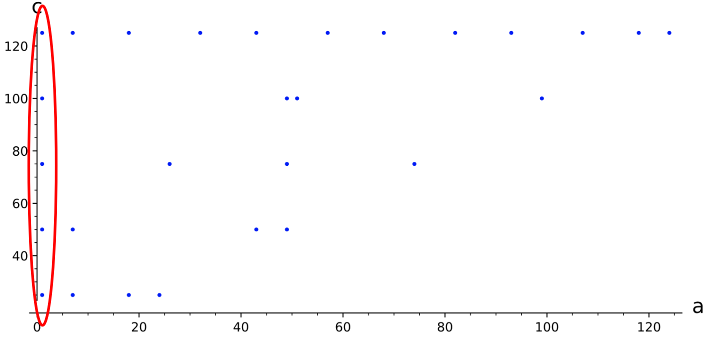

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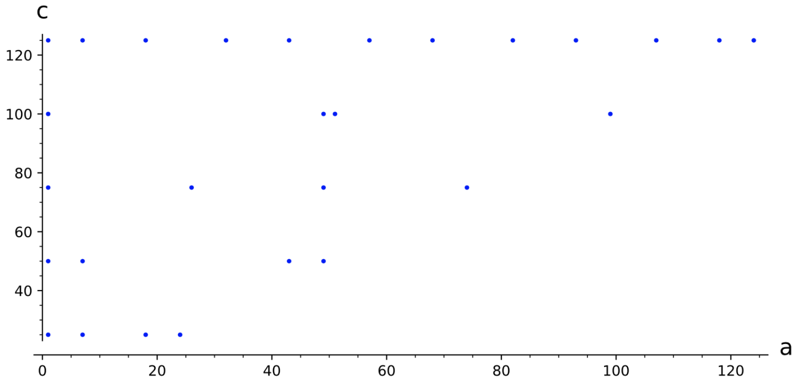

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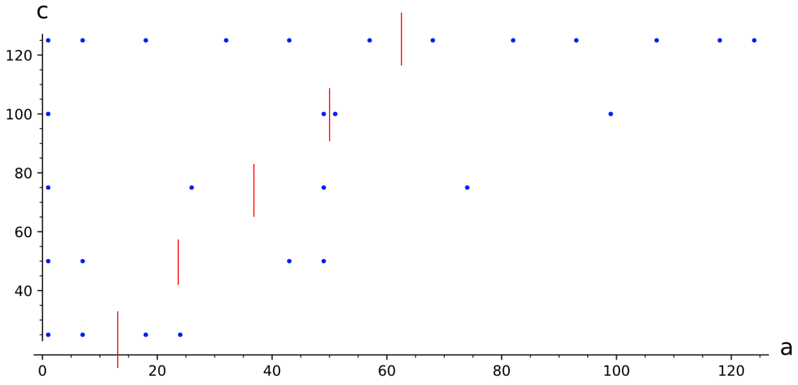

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Definition:

The commutator subgroup C of group G is the subgroup generated by $\{ghg^{-1}h^{-1} : \forall g, h \in G\}$ and is denoted $[G : G]$.

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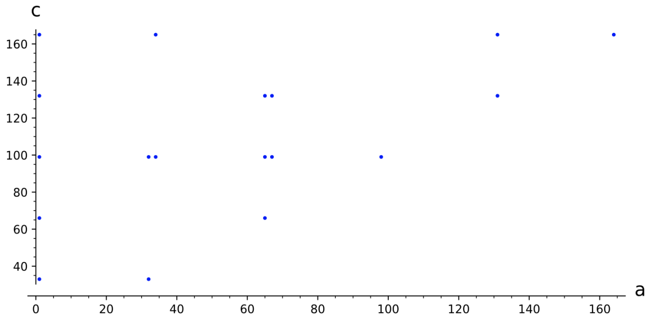
$\{(1, 21), (20, 21), ..., (1, c), (c - 1, c) : c \not\equiv 0 \bmod 441\} \not\subset \Gamma_1(441)$

Identical Kernels

Kernel for $\chi_1 \bmod 3$ & $\chi_3 \bmod 11$ and $\chi_1 \bmod 3$ & $\chi_7 \bmod 11$.



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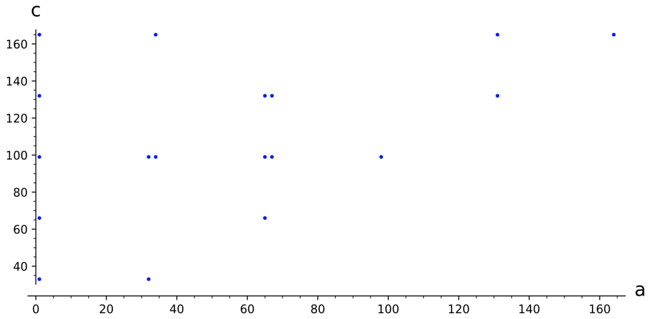

...and also χ_1 mod 3 & χ_9 mod 11 and χ_1 mod 3 & χ_1 mod 11.

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Ex: Image of $\chi_1 \bmod 5$ is $\{0\} \cup \{$4th roots of unity$\}$

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$$\sigma\left(S_{\chi_1,\chi_2}(\gamma)\right) = \sum_{j \pmod c} \sum_{n \pmod{q_1}} \sigma(\overline{\chi_2}(j))\, \sigma(\overline{\chi_1}(n))\, \sigma\left(B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)\right)$$

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Some Notation

Given $q_1, q_2 > 1$, let

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If G acts on S, S breaks up into orbits:
For each $s \in S$,

$$G \cdot s = \{g \cdot s \mid g \in G\}.$$

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Consider the earlier example with mod 3 and mod 11:

- $\chi_1(2) = -1$ and $\chi_1(2) = \zeta_{10}$.
- $\chi_1(2) = -1$ and $\chi_3(2) = \zeta_{10}^3$

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Thank You!

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