Bounds for Coefficients of the $f(q)$ Mock Theta Function and Applications to Partition Ranks (Part 1)

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Partitions

Definition

A **partition** of a positive integer n is a multiset $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ of positive integers such that

- $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$.
- $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

We define the **rank** of this partition as $\lambda_k - k$.

Rank Inequalities

Let $N_e(n)$ and $N_o(n)$ be the number of partitions of n with even and odd rank, respectively.

Conjecture (Hou and Jagadeesan [2], 2017)

- *If* $a, b \geq 11$, *then* $N_e(a)N_e(b) > N_e(a+b)$.
- *If* $a, b \geq 12$, *then* $N_o(a)N_o(b) > N_o(a+b)$.

The $f(q)$ Mock Theta Function

A result of Ramanujan relates the difference of these functions to the Ramanujan mock theta function

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}$$
$$= 1 + \sum_{n=1}^{\infty} (N_e(n) - N_o(n))q^n = \sum_{n=0}^{\infty} \alpha(n)q^n.$$

A remarkable theorem of Bringmann and Ono [4] shows that

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right)$$

However, it is difficult to bound this formula since the sum does **not** converge absolutely.

An Effective Bound for $\alpha(n)$

Theorem (Gomez-Zhu)

Let $D_n = -24n + 1$ *and let* $l(n) = \pi\sqrt{|D_n|}/6$. *Then for all* $n \geq 1$,

$$\alpha(n) = (-1)^{n+1} \frac{\pi}{\sqrt{6} l(n)} e^{l(n)/2} + E(n)$$

where

$$|E(n)| < (4.30 \times 10^{23}) 2^{q(n)} |D_n|^2 e^{l(n)/3}$$

with

$$q(n) := \frac{\log(|D_n|)}{|\log\log(|D_n|) - 1.1714|}.$$

Quadratic Forms

A *positive definite binary quadratic form* is a function of the form $Q(X, Y) = aX^2 + bXY + cY^2$ for integers a, b, c with $a > 0$.

The *discriminant* of this form is $D = b^2 - 4ac$ and we call the form *primitive* if $\gcd(a, b, c) = 1$.

We define $\mathcal{Q}_{D,N}$ as the set of quadratic forms with discriminant $D < 0$, $a \equiv 0 \pmod{N}$ and $b \equiv 1 \pmod{2N}$. Furthermore, we define $\mathcal{Q}_{D,N}^{\text{prim}}$ as the subset of $\mathcal{Q}_{D,N}$ that consists of primitive forms.

Heegner Points

We define the congruence subgroup of $SL_2(\mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{N} \right\}.$$

The group $\Gamma_0(N)$ acts on $\mathcal{Q}_{D,N}$ with the set of orbits $\mathcal{Q}_{D,N}/\Gamma_0(N)$ and similarly acts on $\mathcal{Q}_{D,N}^{\text{prim}}$ with the set of orbits $\mathcal{Q}_{D,N}^{\text{prim}}/\Gamma_0(N)$. The number of orbits in this last set is denoted by the class number $h(D)$.

To each form $Q \in \mathcal{Q}_{D,N}$, we can associate its *Heegner point* τ_Q which is the root of $Q(X, 1)$ with positive imaginary part.

The Brunier-Schwagenscheidt Formula

Consider the modular form:

$$F(z) = q^{-1} - 4 - 83q - 296q^2 + \cdots = \sum_{n=-1}^{\infty} a(n)q^n$$

where $q := e^{2\pi iz}$.

Theorem (Brunier-Schwagenscheidt)

For $n \geq 1$, we have

$$\alpha(n) = -\frac{1}{\sqrt{|D_n|}}\mathrm{Im}(S(n))$$

where

$$S(n) = \sum_{[Q]\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)} F(\tau_Q).$$

Now, we decompose the function into

$$S(n) = \sum_{[Q]\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)} F(\tau_Q) = \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u) \sum_{[Q]\in\mathcal{Q}^{\text{prim}}_{D_n/u^2,6}} F(\gamma_Q(\tau_Q))$$

where $\varepsilon(u) = \pm 1$ and γ_Q are certain right coset representatives of $\Gamma_0(6)$ in $SL_2(\mathbb{Z})$.

Fourier Expansion of $F(z)$

We find the fourier expansion of $F(z)$ as

$$F(\gamma_Q(z)) = \zeta_Q e(-z/h_Q) - 4\beta(h_Q) + \sum_{n=1}^{\infty} \phi_{n,Q} a(n) e(nz/h_Q)$$

where ζ_Q and $\phi_{n,Q}$ are specific sixth roots of unity, $h_Q \in \{1, 2, 3, 6\}$, and $\beta(h_Q) = \pm 1$.

Analyzing $S(n)$

We split this sum up into a main term and an error term as

$$S(n) = \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u) \sum_{Q \in \mathcal{Q}^{\text{prim}}_{D_n/u^2,6}} F(\gamma_Q(\tau_Q))$$
$$= \sum_{Q \in \mathcal{Q}^{\text{prim}}_{D_n,6}} \zeta_Q e(-\tau_Q/h_Q) + E_1(n) + E_2(n).$$

Now, we can analyze the main term, which is a finite sum to get

$$S(n) = (-1)^n i\sqrt{6}\exp(\pi\sqrt{|D_n|}/12) + E_1(n) + E_2(n) + E_3(n).$$

The error terms earlier are given by:

$$E_1(n) := \sum_{\substack{u>1 \\ u^2 | D_n}} \varepsilon(u) \sum_{Q \in \mathcal{Q}^{\text{prim}}_{D_n/u^2, 6}} \zeta_Q e(-\tau_Q/h_Q),$$

$$E_2(n) := 4\beta(h_Q) \sum_{\substack{u>0 \\ u^2 | D_n}} \varepsilon(u) h(D_n/u^2) + \sum_{n=1}^{\infty} \sum_{\substack{u>0 \\ u^2 | D_n}} \varepsilon(u) \phi_{n,Q} a(n) e(n\tau_Q/h_Q),$$

$$E_3(n) := \sum_{\substack{Q \in \mathcal{Q}^{\text{prim}}_{D_n, 6} \\ a_Q h_Q \geq 18}} \zeta_Q e(-\tau_Q/h_Q).$$

The functions $E_1(n)$ and $E_3(n)$ are bounded by the same techniques as for the main term.

Bounding $E_2(n)$

$$E_2(n) := 4\beta(h_Q) \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u) h(D_n/u^2) + \sum_{n=1}^{\infty} \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)\phi_{n,Q} a(n) e(n\tau_Q/h_Q).$$

Proposition (Gomez-Zhu)

For all $n \geq 1$,

$$|a(n)| \leq Ce^{4\pi\sqrt{n}} \text{ where } C := 8\sqrt{6}\pi^{3/2} + 16\pi^2\zeta^2(3/2)$$

Effective Bound for $\alpha(n)$

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$$|E(n)| < (4.30 \times 10^{23}) 2^{q(n)} |D_n|^2 e^{l(n)/3}$$

with

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References

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