Random Interacting Particle Systems and Central Elements of $U(sp_{2n})$

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Probability and Algebra

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Outline

1. What is $U(sp_{2n})$?
2. What is the Random Interacting Particle System?
3. Their connection
4. Methods
5. Results

What is $U(sp_{2n})$?

Definition: sp_{2n}

sp_{2n} is the Lie algebra with elements
$\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} | A = -D^T, B = B^T, C = C^T\}$ where A, B, C, D are $n \times n$ matrices, equipped with Lie bracket $[X, Y] = XY - YX$.

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Basis for sp_{2n}

Let $i, j \in \{\pm 1, \cdots, \pm n\}$. Let E_{ij} denote the matrix with 1 in the (ij)-th entry and 0 everywhere else. Then $F_{ij} = E_{ij} - sgn(ij)E_{-j,-i}$ generate sp_{2n}.

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The universal enveloping algebra of sp_{2n} is
$U(sp_{2n}) = T(sp_{2n})/\langle X \otimes Y - Y \otimes X - [X, Y] \rangle$, where
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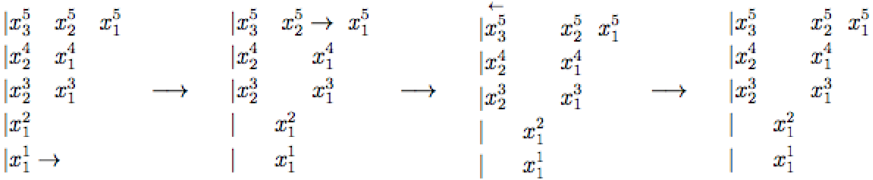

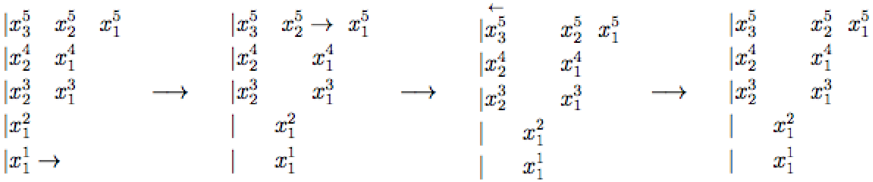

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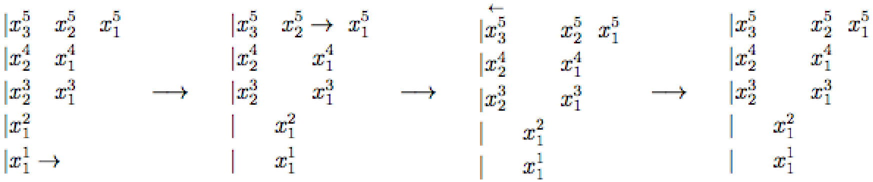

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Theorem (Borodin-Ferrari 2008)

The projection to each level is still a Markov process

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For $\phi_{2k}^N \in Z(U(sp_{2N}))$

$$< \frac{\Phi_{2k}^N}{2} >_{t/2} = \mathrm{E}(p_{2k}^N) \tag{1}$$

- $p_{2k}^N = \sum_{i=1}^{r_n} l_i^{2k}$
- $l_i = \lambda_i - i$, $x_i^N = \lambda_i^N - i + r_n$

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How they work:

We have that $\langle X \rangle_{t+\epsilon} = \langle Q_t X \rangle_\epsilon$

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- After some more math, you find that $\frac{d}{dt} \langle F_{mm}^2 \rangle = Tr(F_{mm}^2) = 2 \implies \langle F_{mm}^2 \rangle_t = 2t$

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- $Cov(p_2^{N_1}(t_1), p_2^{N_2}(t_2)) =$
  $\lim_{L\to\infty} \langle \frac{\Phi_2^{(\eta_1 L)} - \langle \Phi_2^{(\eta_1 L)} \rangle_{\tau_1 L}}{L^2} * \frac{Q_{(\tau_2-\tau_1)L}\Phi_2^{(\eta_2 L)} - \langle Q_{(\tau_2-\tau_1)L}\Phi_2^{(\eta_2 L)} \rangle_{\tau_1 L}}{L^2} \rangle_{\tau_1 L} =$
  $(32\eta_2\eta^2 + 32\eta_1\eta^2 - 32\eta^3)\tau_1 + 64\eta^2\tau_1^2$

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