Reciprocity and the Kernel of Dedekind Sums

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Overview

- Background
  - Dirichlet Characters
  - Eisenstein Series
  - Dedekind Sums
  - $SL_2\mathbb{Z}$

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- Reciprocity
  - The Fricke Involution
  - Reciprocity with Fricke
  - The Atkin-Lehner Involutions
  - Generalized Reciprocity Formula with Atkin-Lehner
  - The effect of the Atkin-Lehner Involutions on Dirichlet Characters

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  - The effect of the Atkin-Lehner Involutions on Dirichlet Characters
- Investigating the Kernel
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Background

Dirichlet Characters

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- *If* $gcd(n, k) = 1$*, then* $\chi(n) \neq 0.$

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- $\chi(1) = 1$.

Note that χ is *even* if $\chi(-1) = 1$ and χ is *odd* if $\chi(-1) = -1$.

Let χ_1, χ_2 be primitive Dirichlet characters with conductors q_1, q_2 respectively. The weight-zero Eisenstein Series of $z \in \mathbb{C}$ associated with Dirichlet characters χ_1 and χ_2 is as follows:

Eisenstein Series

$$E_{\chi_1,\chi_2}(z,s) = \frac{1}{2} \sum_{(m,n)=1} \frac{(q_2 y)^s \chi_1(m)\chi_2(n)}{\mid mq_2 z + n \mid^{2s}}, \quad Re(s) > 1$$

- Through the Dedekind η-function, Eisenstein series give rise to certain Dedekind Sums

Dedekind Sums

The classical Dedekind Sum $S_{\chi_1,\chi_2}(\gamma)$ is defined as follows:

Dedekind Sum

$$S_{\chi_1,\chi_2}(\gamma) = \frac{\tau(\overline{\chi_1})}{\pi i}\phi_{\chi_1,\chi_2}(\gamma),$$

where $\gamma \in \Gamma_0(q_1 q_2)$ and $\phi_{\chi_1,\chi_2}(\gamma) = f_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) f_{\chi_1,\chi_2}(z)$.

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($f_{\chi_1,\chi_2}(z)$ arises from the Fourier expansion of the completed Eisenstein series)

$$E_{\chi_1,\chi_2}(\gamma z) = \psi(\gamma) E_{\chi_1,\chi_2}(z)$$

$$\psi(\gamma) = \chi_1(d)\overline{\chi_2}(d)$$

$$S_{\chi_1,\chi_2} : SL_2\mathbb{Z} \to \mathbb{H}$$

$$SL_2\mathbb{Z} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}; ad - bc = 1 \right\}.$$

- $\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \;\middle|\; c \equiv 0 \pmod{q} \right\}.$
- $\Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \;\middle|\; a \equiv d \equiv 1 \pmod{q}; c \equiv 0 \pmod{q} \right\}.$
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Reciprocity

The Fricke Involution

$$\omega = \omega_{q_1 q_2} = \begin{pmatrix} 0 & -1 \\ q_1 q_2 & 0 \end{pmatrix}$$

- The Eisenstein series is a pseudo-eigenfunction of the Fricke involution:
  - $E_{\chi_1,\chi_2}(\omega z, s) = \chi_2(-1) E_{\chi_1,\chi_2}(z, s)$
- The Fricke involution swaps the characters associated to the Dedekind sum; χ_1 becomes χ_2 and vice versa

Reciprocity with Fricke

Theorem (SVY)

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$, *let* $\gamma' = \begin{pmatrix} d & -c \\ -bq_1 q_2 & a \end{pmatrix} \in \Gamma_0(q_1 q_2)$. *If* χ_1 *and* χ_2 *are even, then*

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_2,\chi_1}(\gamma').$$

If χ_1 *and* χ_2 *are odd, then*

$$S_{\chi_1,\chi_2}(\gamma) = -S_{\chi_2,\chi_1}(\gamma').$$

The Atkin-Lehner Involutions

The Fricke Involution

$$\omega = \omega_{q_1 q_2} = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

The Fricke involution is associated to some N.
Let $N = p_1^{q_1} *. \ . \ .* \ p_{r^{q_r}}$ be the prime factorization of N. There is an Atkin-Lehner involution ω_{p_r} associated to each prime factor p_r of N.

The Atkin-Lehner Involution

Definition

Suppose that $QR = N$ *and* $(Q, R) = 1$. *We define an Atkin-Lehner operator by*

$$W_Q = \begin{pmatrix} Qr & t \\ Nu & Qv \end{pmatrix},$$

where $r, t, u, v \in \mathbb{Z}$, $r \equiv r_0 \pmod{R}$ *and* $t \equiv t_0 \pmod{Q}$ *such that* $Qrv - Rut = 1$.

As the Atkin-Lehner involutions form a family of operators closely connected to the Fricke involution, we found that the reciprocity formulas of these Dedekind sums form a **family** of formulas, one for each Atkin-Lehner involution,

Generalized Reciprocity Formula with Atkin-Lehner

Let χ_1, χ_2 be primitive Dirichlet characters with moduli q_1, q_2, respectively. The following theorem holds for any Atkin-Lehner involution W_Q and W_Q' such that $W_Q\gamma = \gamma' W_Q'$, and $\gamma, \gamma' \in \Gamma_0(q)$.

Theorem

$$S_{\chi_1,\chi_2}(W_Q) + \xi S_{\chi_1'\chi_2'}(\gamma) = \overline{\psi}(\gamma) S_{\chi_1',\chi_2'}(W_Q') + S_{\chi_1,\chi_2}(\gamma'),$$

where $\xi = \frac{q_2 \tau(\chi_2')}{q_2' \tau(\chi_2)} \chi_2^{(Q)}(-1) \overline{\psi}^{(Q)}(q_2^{(R)} t_0)) \overline{\psi}^{(R)}(q_2^{(Q)} r_0))$

and $\overline{\psi}(\gamma) = \chi_1' \overline{\chi_2'}$

Let χ_1, χ_2 be primitive Dirichlet characters with moduli q_1, q_2, respectively. The following theorem holds for any Atkin-Lehner involution W_Q and W'_Q such that $W_Q\gamma = \gamma' W'_Q$, and $\gamma, \gamma' \in \Gamma_0(q)$.

Theorem

$$S_{\chi_1,\chi_2}(W_Q) + \xi S_{\chi'_1\chi'_2}(\gamma) = \overline{\psi}(\gamma) S_{\chi'_1,\chi'_2}(W'_Q) + S_{\chi_1,\chi_2}(\gamma'),$$

where $\xi = \frac{q_2\tau(\chi'_2)}{q'_2\tau(\chi_2)}\chi_2^{(Q)}(-1)\overline{\psi}^{(Q)}(q_2^{(R)}t_0))\overline{\psi}^{(R)}(q_2^{(Q)}r_0))$

and $\overline{\psi}(\gamma) = \chi'_1\overline{\chi'_2}$

If $W_Q = (W_Q)'$, the formula simplifies as

$$S_{\chi_1,\chi_2}(\gamma') = (1 - \overline{\psi}(\gamma))S_{\chi_1,\chi_2}(W_Q) + \xi S_{\chi'_1\chi'_2}(\gamma).$$

Atkin-Lehner Involutions and Dirichlet Characters

Fricke Involution ω:

- $\chi_1 \to \chi_2$
- $\chi_2 \to \chi_1$

Atkin-Lehner Involution W_Q associated to prime factor Q:

*Recall $q_1 q_2 = N = QR$

- $\chi_1 = \chi_1^{(Q)}\chi_1^{(R)} \to \chi_2^{(Q)}\chi_1^{(R)}$
- $\chi_2 = \chi_2^{(Q)}\chi_2^{(R)} \to \chi_1^{(Q)}\chi_2^{(R)}$

The effect of Atkin-Lehner on Dirichlet Characters

$\chi_1' = \chi_2^{(Q)}\chi_1^{(R)}$ and $\chi_2' = \chi_1^{(Q)}\chi_2^{(R)}$

Investigating the Kernel

Let χ_1, χ_2 be primitive Dirichlet characters with conductors q_1, q_2 respectively, with $q_1, q_2 > 1$. Then the kernel of the Dedekind sum $S(h,k)$ associated to χ_1, χ_2 is defined by:

Kernel associated to χ_1, χ_2

$$K_{\chi_1,\chi_2} = ker(S_{\chi_1,\chi_2}) = \{\gamma \in \Gamma_0(q_1 q_2) \mid S_{\chi_1,\chi_2}(\gamma) = 0\}$$

If $\overline{\psi}(\gamma) = 1$, the reciprocity formula simplifies to:

$$S_{\chi_1,\chi_2}(\gamma') = \xi S_{\chi_1' \chi_2'}(\gamma)$$

So, $\gamma' \in K_{\chi_1,\chi_2} \iff \gamma \in K_{\chi_1',\chi_2'}$.

Recall $W_Q \gamma = \gamma' W_Q$. So $\gamma = W_Q^{-1} \gamma' W_Q$.

$$\gamma' \in K_{\chi_1,\chi_2} \iff W_Q^{-1} \gamma' W_Q \in K_{\chi_1',\chi_2'}$$

Definition

$$S_{\chi_1,\chi_2}(\gamma) = \sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right) \text{ where}$$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$ *with* $c \geq 1$ *and* $\chi_1\chi_2(-1) = 1$.

B_1 *is the first Bernoulli function defined by*

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & if x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & if x \in \mathbb{Z}. \end{cases}$$

Dedekind Sums and Elements of K_{χ_1,χ_2}

Definition

$$S_{\chi_1,\chi_2}(\gamma) = \sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right) \text{ where}$$

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The value of $S_{\chi_1,\chi_2}(\gamma)$ solely depends on the first column of γ, so we are allowed to use the equivalent notation $S_{\chi_1,\chi_2}(a, c)$.

Known Kernel Elements

Proposition (Nguyen, Ramirez, Young)

$S_{\chi_1,\chi_2}(1, c'q_1q_2) = 0$ *for all* $c' \in \mathbb{Z}$

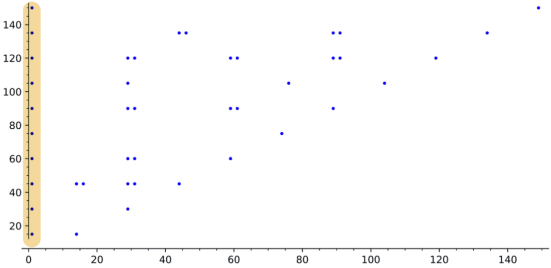

Figure: $K_{3,5}$ for $1 \leq c \leq 10q_1q_2$

Known Kernel Elements

Proposition (Nguyen, Ramirez, Young)

For every (a, c) in the kernel, $(c - a, c)$ is also in the kernel.

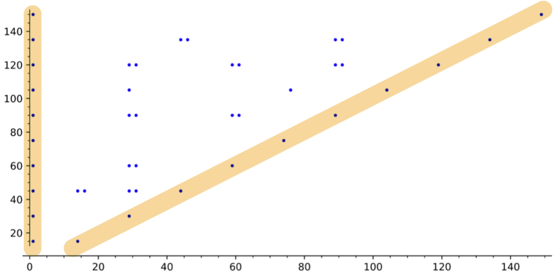

Figure: $K_{3,5}$ for $1 \leq c \leq 10 q_1 q_2$

General Formula for Kernel Elements from Atkin-Lehner Involutions

Theorem

Let χ_1 and χ_2 be nontrivial primitive Dirichlet characters modulo q_1, q_2, respectively. Let $W_Q = \begin{pmatrix} Qr & t \\ Nu & Qv \end{pmatrix}$ be an Atkin-Lehner operator. Then $S_{\chi_1', \chi_2'}(1 - Ntkr, QNkr^2) = 0$ for all $k \in \mathbb{Z}$.

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Overview of Proof. We take $\gamma' = \begin{pmatrix} 1 & 0 \\ kq_1q_2 & 1 \end{pmatrix}$. Rearranging the relationship $W_Q\gamma = \gamma' W_Q$ from our reciprocity formula gives

$$\gamma = (W_Q)^{-1}\gamma' W_Q = \begin{pmatrix} 1 - Ntkr & Ntkr \\ QNkr^2 & 1 + Ntkr \end{pmatrix}.$$

We see that since $\gamma' \in K_{\chi_1, \chi_2}$, $\gamma \in K_{\chi_1', \chi_2'}$. Thus, for all $k \in \mathbb{Z}$, $S_{\chi_1', \chi_2'}(1 - Ntkr, QNkr^2) = 0$, as desired.

General Formula for Kernel Elements from Atkin-Lehner Involutions

Proposition

Let χ_1 and χ_2 be nontrivial primitive Dirichlet characters modulo q_1, q_2, respectively. Let $W_Q = \begin{pmatrix} Qr & t \\ Nu & Qv \end{pmatrix}$ *be an Atkin-Lehner operator. Then* $S_{\chi_1', \chi_2'}(-1 - Ntkr, QNkr^2) = 0$ *for all* $k \in \mathbb{Z}$.

General Formula for Kernel Elements from Atkin-Lehner Involutions

Proposition

Let χ_1 and χ_2 be nontrivial primitive Dirichlet characters modulo q_1, q_2, respectively. Let $W_Q = \begin{pmatrix} Qr & t \\ Nu & Qv \end{pmatrix}$ be an Atkin-Lehner operator. Then $S_{\chi_1', \chi_2'}(-1 - Ntkr, QNkr^2) = 0$ for all $k \in \mathbb{Z}$.

Note. An easy modification of the proof of our last theorem using $\gamma' = \begin{pmatrix} -1 & 0 \\ kq_1q_2 & -1 \end{pmatrix}$ completes the proof.

Elements of the Kernel

Corollary

The kernel includes all pairs of elements $(\pm 1 + Nk, QNk)$ and $(\pm 1 + (Q-1)Nk, QNk)$

Overview of Proof. Let the Atkin-Lehner operator W_Q be such that $r = 1$, $t = 1$. Then by the previous theorem,

$$S_{\chi_1', \chi_2'}(1 - Ntkr, QNkr^2) = S_{\chi_1', \chi_2'}(1 - Nk, QNk) = 0.$$

Using properties from SVY, it follows that

$$S_{\chi_1', \chi_2'}(1 + (Q-1)Nk, QNk) = 0 \text{ and } S_{\chi_1', \chi_2'}(-1 + Nk, QNk) = 0$$

Similarly, by the analogous proposition, $S_{\chi_1', \chi_2'}(-1 - Nk, QNk) = 0$. Then, using properties from SVY, it follows that

$$S_{\chi_1', \chi_2'}(-1 + (Q-1)Nk, QNk) = 0 \text{ and } S_{\chi_1', \chi_2'}(1 + Nk, QNk) = 0.$$

Altogether, these symmetries explain the pairs of kernel elements $(\pm 1 + Nk, QNk)$ and $(\pm 1 + (Q-1)Nk, QNk)$.

Example $K_{3,5}$. $N = 15$, $Q = 3$, $R = 5$

Our Atkin-Lehner matrix $W_3 = \begin{pmatrix} 3 & 1 \\ 15 & 6 \end{pmatrix}$. We calculate

$$(W_3)^{-1}\gamma' W_3$$

with $k = 1$ and

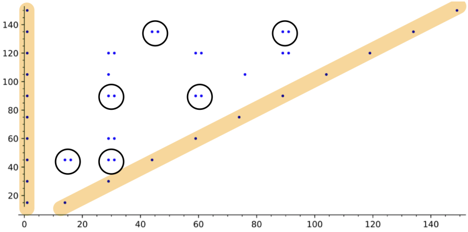
$$\gamma' = \begin{pmatrix} 1 & 0 \\ kq_1q_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 15 & 1 \end{pmatrix}.$$

We obtain the product

$$\begin{pmatrix} -14 & -5 \\ 45 & 16 \end{pmatrix}.$$

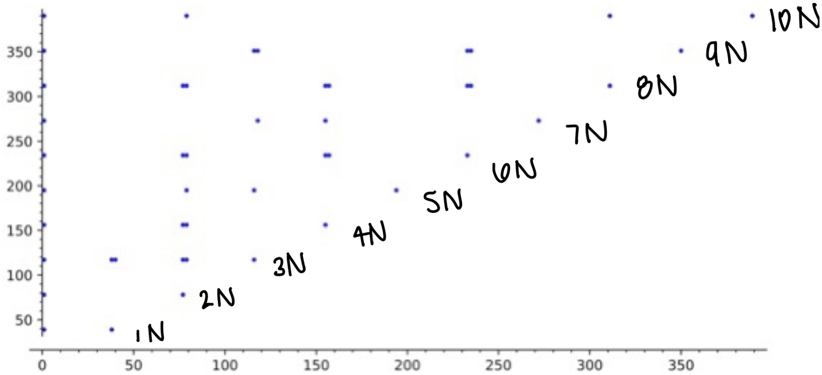
Example $K_{3,5}$. $N = 15,\ Q = 3,\ R = 5$

Our product was $\begin{pmatrix} -14 & -5 \\ 45 & 16 \end{pmatrix}$.



- $(a, c) = (-14, 45)$
- Looking $a \pmod{c}$, we obtain $(31, 45)$
- $(c - a, c) = (14, 45)$,
- By our proposition, we obtain $(16, 45)$ and $(29, 45)$

Terminology Moving Forward



Example: $(\pm 1 + tN, QN)$

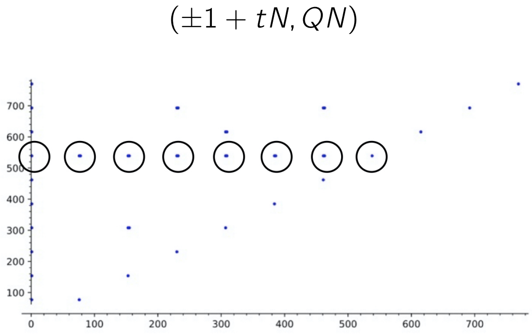


Figure: $K_{7,11}$ for $1 \le c \le 10q_1q_2$

Example: $(\pm 1 + tkN, t^2kN)$

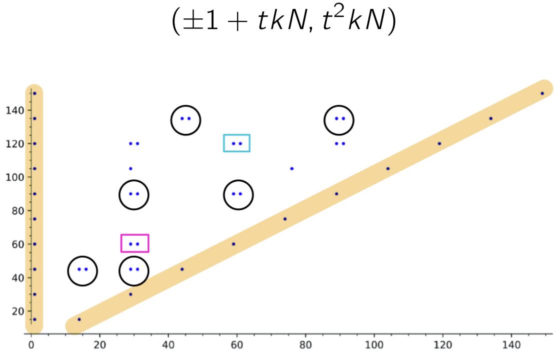


Figure: $K_{3,5}$ for $1 \leq c \leq 10q_1q_2$

Future Study

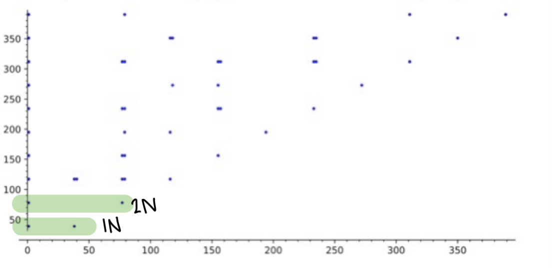


Figure: $K_{3,13}$ for $1 \leq c \leq 10q_1q_2$

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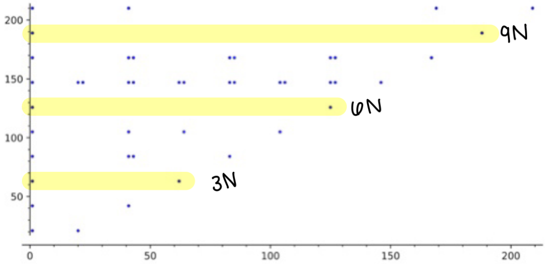

Figure: $K_{7,3}$ for $1 \leq c \leq 10q_1q_2$

Future Study

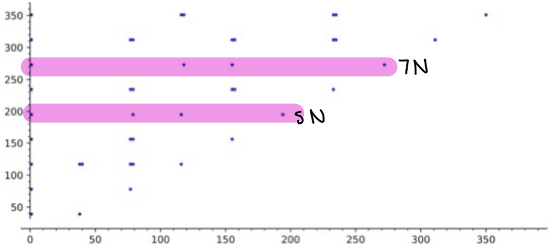

Figure: $K_{3,13}$ for $1 \leq c \leq 10q_1q_2$

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