A THREE TERM RELATION FOR NEWFORM DEDEKIND SUMS

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Abstract. For the classical Dedekind sum, Dedekind's reciprocity relation and Rademacher's three term relation are well-known. For newform Dedekind sums, Stucker, Vennos, and Young proved a "two term" reciprocity relation. In this paper, we prove a three term reciprocity relation analogous to Rademacher's formula. We also prove a three term relation involving six variables which is analogous to a result of Pommersheim.

1. Introduction

1.1. Classical Dedekind Sums.

The classical Dedekind sum, is defined for integers a, c as

$$s(a,c) = \sum_{n \bmod c} B_1\left(\frac{n}{c}\right) B_1\left(\frac{an}{c}\right)$$

where

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \text{ and } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

There exists a "two term" reciprocity relation such that for relatively prime integers a and c,

$$s(a,c) + s(c,a) = \frac{1}{12}\left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a}\right) - \frac{1}{4}. \tag{1}$$

The reciprocity formula is a fundamental relation in number theory. It has been used to systematically count integer points in polytopes and to efficiently compute the Dedekind sum. Rademacher deduced the following "three term" reciprocity relation between Dedekind sums.

Theorem 1.1 (Rademacher)**.** *For mutually coprime natural numbers p, q and r,*

$$s(p\overline{q}, r) + s(q\overline{r}, p) + s(r\overline{p}, q) = \frac{1}{12}\left(\frac{p}{qr} + \frac{q}{pr} + \frac{r}{pq}\right) - \frac{1}{4}.$$

where $\overline{q}$ and $\overline{r}$, $\overline{p}$ are the inverses of $r \pmod p$ and $p \pmod q$, respectively.

Rademacher's result easily implies (1). Rademacher did not deduce this "three term" relation from the two term and believed that the three term "seemed to go beyond" its properties [Rad54, 391]. This relation is a useful tool in [Gir21] for studying the equality of two Dedekind sums. Additionally it is used to help determine the mean value of Dedekind sums in [CFKS96]. Several authors have also given generalizations of Dedekind sums and in turn proved associated reciprocity relationships. The authors in [HWZ95] considered a generalization in terms of higher order Bernoulli functions and proved a three term relationship for a related function utilizing Theorem 1.1.

Pommersheim also proved a relationship between three Dedekind sums in six variables. He obtained this result for Dedekind sums while studying the relationships between lattice points, the Todd class of toric varieties, and how Dedekind sums can be used in related computations [Pom93].

Theorem 1.2 (Pommersheim)**.** *Let p, q, u, v be natural numbers with $(p,q) = (u,v) = 1$ and u^*, v^* be integers such that $uu^* + vv^* = 1$. Define x and y by $x = qv^* - pu^*$ and $y = pv + qu$. Then*

$$s(p,q) + s(u,v) + s(x,y) = \frac{1}{12}\left(\frac{q}{vy} + \frac{v}{qy} + \frac{y}{qv}\right) - \frac{1}{4}.$$

Pommersheim classified this relationship as a generalization of Theorem 1.1.

1.2. **Newform Dedekind Sums.** In this section we give the definition of the newform Dedekind sum obtained in [SVY20] as well as some of the properties proved within the paper.

Definition 1.3. The group $\Gamma_0(N)$, $N \in \mathbb{Z}$, is defined as

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}$$

Definition 1.4 (Newform Dedekind Sums)**.** Let $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ be primitive Dirichlet characters with $q_1, q_2 > 1$ and $\chi_1\chi_2(-1) = 1$. Let $c \geq 1$ and

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(q_1q_2).$$

Then

$$S_{\chi_1\chi_2}(\gamma) = \sum_{j \,(\mathrm{mod}\ c)} \sum_{n \,(\mathrm{mod}\ q_1)} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right).$$

Additionally, they found the following two term reciprocity relation in the newform case.

Theorem 1.5 ([SVY20])**.** *Let* $\gamma = \begin{bmatrix} a & b \\ cq_1q_2 & d \end{bmatrix} \in \Gamma_0(q_1q_2)$ *and* $\gamma' = \begin{bmatrix} d & -c \\ -bq_1q_2 & a \end{bmatrix} \in \Gamma_0(q_1q_2)$. *Then for* $\chi_1,\ \chi_2$ *even,*

$$S_{\chi_1,\chi_2}(\gamma) = S_{\chi_2,\chi_1}(\gamma').$$

Remark. A reciprocity formula for $\chi_1,\ \chi_2$ odd also appears in [SVY20].

Remark. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we write $S_{\chi_1,\chi_2}(\gamma) = S_{\chi_1,\chi_2}(a, c)$ which mimics the notation in the classical case.

The definition of the newform Dedekind sum can be extended to $c = 0$ and $c < 0$ via $S_{\chi_1\chi_2}(-a, -c) = S_{\chi_1\chi_2}(a, c)$ and $S_{\chi_1\chi_2}(1, 0) = 0$.

The first result in our paper is an analogue of Theorem 1.2 in the newform case.

Theorem 1.6. *Let* p, q, u, v *be natural numbers with* $(p, q) = (u, v) = 1$, $q_1q_2|v$, *and* $q_1q_2|q$, *and* u^*, v^* *be integers such that* $uu^* + vv^* = 1$. *Define* x *and* y *by* $x = qv^* + pu^*$ *and* $y = -pv + qu$. *Then*

$$S_{\chi_1,\chi_2}(p, q) - S_{\chi_1,\chi_2}(u, v) - \overline{\psi}(u) S_{\chi_1,\chi_2}(x, y) = 0.$$

Theorem 1.6 appears slightly different from Theorem 1.2 because of our different definitions of x and y.

The second theorem we prove is an analogue in the newform case for Theorem 1.1.

Theorem 1.7. *Suppose* Q, V, Y *are pairwise coprime integers. Moreover, suppose* $\gcd(QVY, q_1q_2) = 1$. *Define integers* P, U, *and* X *by the following congruences*

$$P \equiv \begin{cases} \overline{Y}V & \pmod{Q} \\ 1 & \pmod{q_1q_2} \end{cases}, \quad U \equiv \begin{cases} Y\overline{Q} & \pmod{V} \\ 1 & \pmod{q_1q_2} \end{cases}, \quad X \equiv \begin{cases} Q\overline{V} & \pmod{Y} \\ 1 & \pmod{q_1q_2} \end{cases}. \tag{2}$$

Then

$$S_{\chi_1,\chi_2}(P, Qq_1q_2) + S_{\chi_1,\chi_2}(U, Vq_1q_2) + S_{\chi_1,\chi_2}(X, Yq_1, q_2) = 0.$$

1.3. **Discussion.** In [Gir99], Girstmair showed that Rademacher's three term relation is a consequence of Pommersheim's relation, which is itself a consequence of the famous two term reciprocity relation. That is, he proved a direct equivalence between the three relations. Within his proof Girstmair used Theorem 1.2 as an intermediary step between the classical and Rademacher reciprocity relations although Pommersheim's result was originally conceived as a generalization of Theorem 1.1. Although in the classical case there is a direct equivalence between the two term and the three term relations, we did not find any such equivalence in the newform case between our two results and the two term reciprocity relation in [SVY20].

2. Properties of Newform Dedekind Sums

The following properties of newform Dedekind sums are used throughout the paper.

Lemma 2.1. *For $\gamma_1,\ \gamma_2 \in \Gamma_0(q_1q_2)$,*

$$S_{\chi_1,\chi_2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)S_{\chi_1,\chi_2}(\gamma_2).$$

Proof. See [SVY20, Lemma 2.2]. □

Lemma 2.2. *Let $\gamma \in \Gamma_0(q_1q_2)$ and $\gamma^{-1} \in \Gamma_0(q_1q_2)$ be its multiplicative inverse. Then*

$$S_{\chi_1,\chi_2}(\gamma) = -\psi(\gamma)S_{\chi_1,\chi_2}(\gamma^{-1}).$$

Proof. This follows from Lemma 2.1. □

Lemma 2.3. *For $a, c \in \mathbb{Z}$, with $c \geq 1$ and $q_1q_2|c$,*

$$S_{\chi_1,\chi_2}(-\overline{a}, c) = -\psi(a)S_{\chi_1,\chi_2}(a, c).$$

Proof. Follows from [DG20, Propositions 2.3, 2.4] □

3. Proof of Theorem 1.6

Let p, q, u, v be natural numbers with $(p, q) = (u, v) = 1$, $q_1q_2|v$, and $q_1q_2|q$, and u^*, v^* be integers such that $uu^* + vv^* = 1$. Define x and y by $x = qv^* + pu^*$ and $y = -pv + qu$. Then

$$M_1 = \begin{bmatrix} u^* & v^* \\ -v & u \end{bmatrix} \quad \text{and} \quad {M_1}^{-1} = \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix}. \tag{3}$$

$$M_2 = \begin{bmatrix} p & -q^* \\ q & p^* \end{bmatrix}. \tag{4}$$

Note that $M_1,\ M_2 \in \Gamma_0(q_1q_2)$.

Define the matrix $M_3 \in \Gamma_0(q_1q_2)$ such that $M_1M_2M_3 = I$ where I is the identity matrix. By direct calculation,

$$M_3 = (M_1M_2)^{-1} = \begin{bmatrix} vq^* + p^*u & q^*u^* - v^*p^* \\ pv - qu & pu^* + qv^* \end{bmatrix} = \begin{bmatrix} * & * \\ -y & x \end{bmatrix} \quad \text{and} \quad {M_3}^{-1} = \begin{bmatrix} x & * \\ y & * \end{bmatrix}. \tag{5}$$

On the one hand,

$$S_{\chi_1,\chi_2}(M_1M_2M_3) = S_{\chi_1,\chi_2}(I) = 0. \tag{6}$$

By Lemma 2.1,

$$0 = S_{\chi_1,\chi_2}(M_1) + \psi(M_1)S_{\chi_1,\chi_2}(M_2) + \psi(M_1M_2)S_{\chi_1,\chi_2}(M_3). \tag{7}$$

On the other hand, using Lemma 2.2 to invert M_1 and M_3, we have,

$$\begin{aligned} 0 &= -\psi(M_1)S_{\chi_1,\chi_2}({M_1}^{-1}) + \psi(M_1)S_{\chi_1,\chi_2}(M_2) - \psi(M_1M_2)\psi(M_3)S_{\chi_1,\chi_2}({M_3}^{-1}) \\ &= -\psi(M_1)S_{\chi_1,\chi_2}({M_1}^{-1}) + \psi(M_1)S_{\chi_1,\chi_2}(M_2) - S_{\chi_1,\chi_2}({M_3}^{-1}). \end{aligned} \tag{8}$$

Multiplying each term by $\overline{\psi}(M_1)$, we obtain

$$0 = -S_{\chi_1,\chi_2}({M_1}^{-1}) + S_{\chi_1,\chi_2}(M_2) - \overline{\psi}(M_1)S_{\chi_1,\chi_2}({M_3}^{-1}). \tag{9}$$

Using the notation we defined in our introduction, we get Theorem 1.6.

4. **Proof of Theorem 1.7**

Let $Q, V, Y \in \mathbb{Z}$ and q_1q_2 be mutually coprime. Set q, v, and y as $q = q_1q_2Q, v = q_1q_2V, y = q_1q_2Y$. Select integers P, U, and X such that (2) holds.
This implies that

$$UQ \equiv Y \pmod{V}.$$

Following from $V, Y|q_1q_2$, let there be some integer L such that

$$UQ = Y + VL.$$

Then

$$Y \equiv -VL \pmod{Q}.$$

So,

$$L \equiv -Y\overline{V} \pmod{Q}.$$

That is,

$$L \equiv -\overline{P} \pmod{Q}.$$

Let $u = U$ and let $u^*, v^* \in \mathbb{Z}$ be such that $uu^* + vv^* = 1$. Set $p \in \mathbb{Z}$ such that $p \equiv \begin{cases} L \pmod{Q} \\ -1 \pmod{q_1q_2} \end{cases}$ and let $p^*, q^* \in \mathbb{Z}$ be such that $pp^* + qq^* = 1$.
Note that

$$uQ = Y + VL \Longleftrightarrow uq = y + vL.$$

Let x, y be as defined in Theorem 1.6. This implies

$$xv = pu^*v + q(1 - uu^*)$$

$$= q + u^*(pv - qu)$$

$$= q - u^*y.$$

Thus

$$xV = Q - u^*Y$$

and so

$$x \equiv Q\overline{V} \equiv X \pmod{Y}.$$

With u, v, p, q, x, y defined above, let $M_1M_2M_3$ be given by (3), (4), and (5). Then $\psi(u) = 1$.
The proof of Theorem 1.6 gives

$$S_{\chi_1,\chi_2}(L, q_1q_2Q) + S_{\chi_1,\chi_2}(U, q_1q_2V) + S_{\chi_1,\chi_2}(X, q_1, q_2Y) = 0$$

Using Lemma 2.3 we obtain Theorem 1.7. When $q_1q_2 = 1$ this formula reduces down to

$$S(\overline{Y}V, Q) + S(\overline{Q}Y, V) + S(\overline{V}Q, Y) = 0.$$

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References

[DG20] T Dillon and S Gaston, *An average of generalized Dedekind sums*, J. of Number Theory **212** (2020), 323–338, DOI https://doi.org/10.1016/j.jnt.2019.11.008.

[Gir99] Kurt Girstmair, *Some remarks on Rademacher's three-term relation*, Arch. Math. (Basel) **73** (1999), no. 3, 205–207, DOI 10.1007/s000130050029. MR1705016

[SVY20] T. Stucker, A. Vennos, and M. P. Young, *Dedekind sums arising from newform Eisenstein series*, Int. J. Number Theory **16** (2020), no. 10, 2129–2139, DOI 10.1142/S1793042120501092. MR4167772

[Rad54] Hans Rademacher, *Generalization of the reciprocity formula for Dedekind sums*, Duke Math. J. **21** (1954), 391–397. MR62765

[Pom93] James E. Pommersheim, *Toric varieties, lattice points and Dedekind sums*, Math. Ann. **295** (1993), no. 1, 1–24, DOI 10.1007/BF01444874. MR1198839

[Gir21] Kurt Girstmair, *On the equality of Dedekind sums*, Integers **21** (2021), Paper No. A98, 9. MR4328204

[CFKS96] J. B. Conrey, Eric Fransen, Robert Klein, and Clayton Scott, *Mean values of Dedekind sums*, J. Number Theory **56** (1996), no. 2, 214–226, DOI 10.1006/jnth.1996.0014. MR1373548

[HWZ95] R. R. Hall, J. C. Wilson, and D. Zagier, *Reciprocity formulae for general Dedekind-Rademacher sums*, Acta Arith. **73** (1995), no. 4, 389–396, DOI 10.4064/aa-73-4-389-396. MR1366045

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