Goldbach’s Conjecture

by James Davis

July 19, 2006

Abstract

Goldbach’s Conjecture, that every number can be written as a sum of two primes, has been neither proven nor disproven since it was posited in the 1740’s. The most recent research into the topic (by Oliveira e Silva in 2005) showed that every even number less than \(3 \times 10^{17}\) can be written as a sum of two primes. This research shows that there are large numbers of even numbers, (some possibly beyond this limit) for which we can also easily confirm as being the sum of two primes. While this is not a complete proof of Goldbach’s Conjecture, it does confirm that it holds locally at a large number of \(n\).
1 Current Research

Before we continue, there are two important definitions we need to posit.

Definition 1. Given a number, \( n \), we may say that any number \( m \) less than \( n \) is a totive of \( n \) iff \( m \nmid n \).

Definition 2. \( \phi_C(n) \) is the number of composite totives of \( n \).

Theorem 3. \( \phi_C(n) < \pi(n) \) implies that \( n \) can be written as a sum of two primes.

Proof. In general, even numbers that can be written as the sum of two primes do not have a unique way of doing so; therefore, it is useful to define an algorithm for writing such an algorithm.

Let \( \mathbb{P} \) be the set of all primes.

Now define the sets,

\[
\mathbb{P}_0(n) := \{ p \in \mathbb{P} \mid p < n \}
\]

\[
\mathbb{P}_N(n) := \mathbb{P}_{N-1}(n) - \{ \max \{ \mathbb{P}_{N-1}(n) \} \} \forall N \geq 1
\]

Now find

\[
m := \min \{ \{ N \in \mathbb{N} \mid \max \{ \mathbb{P}_m(n) \} \in \mathbb{P} \} \}
\]

\[
n = \max \{ \mathbb{P}_m(n) \} + (n - \max \{ \mathbb{P}_m(n) \})
\]

Suppose there exists a \( D \) that cannot be written as a sum of two primes. Then

\[
\forall n \in \mathbb{N} \left( D - \max \{ \mathbb{P}_m(D) \} \right) \notin \mathbb{P}
\]

Then clearly for any prime \( p_i < D \), \( D - p_i \) is a composite number.

Another form of this assertion is that for any \( D \) that cannot be written as the sum of two primes, for every prime \( p_i \) there exists a composite \( c_i \) (both of which are less than \( D \)), such that

\[
D = p_i + c_i
\]

We may observe that \( p_i + c_i = D \) implies that \( p_i = D - c_i \). Now, if \( D \) and \( c_i \) shared a common factor (Say, for example, \( D = \kappa d \) and \( c_i = \kappa k_i \)), then we could write

\[
p_i = \kappa d - \kappa k_i = \kappa (d - k_i)
\]

But this implies that \( p_i \) is not prime.

Therefore, \( c_i \) and \( N \) can have no common factors (Because \( c_i \) is less than \( N \), we can say that \( c_i \) is a totive of \( N \)). Therefore, this implies that any \( D \) (even) that cannot be written as a sum of two primes must satisfy:

\[
\phi_C(D) \geq \pi(D)
\]

If \( D \) did not satisfy this, then there would not be enough possible \( c_i \) to go with every \( p_i \). Therefore, any number for which the negation of (8) is true can be written as a sum of two primes. \( \Box \)
2 Future Research

Unfortunately, this relationship does not prove Goldbach’s Conjecture, it only shows that any ‘disproofs’ will likely be isolated cases.

Research is continuing towards the following:

**Theorem 4.** There exists no $n$ such that $\phi_C(n + m) \geq \pi(n + m)$ for all $m \in \mathbb{N}$.

This appears to be true because we can construct large $k$ with very low $\phi_C(k)$, but I am not certain I have obtained a proof at this time.

Proof of theorem 4 would show that there are infinitely many even numbers that can be written as a sum of two primes.

**Theorem 5.** Goldbach’s Conjecture

It is not hard to see that because of the relation $D = p_i + c_i$ we are guaranteed that if $D$ cannot be written as a sum of two primes either $p_i < \frac{D}{2}$ and $c_i > \frac{D}{2}$ or $p_i > \frac{D}{2}$ and $c_i < \frac{D}{2}$.

Because primes are not distributed symmetrically about $\frac{D}{2}$ this approach (and perhaps one with slightly smaller bounds) could show that even if $\phi_C(D) \geq \pi(D)$ the totives of $D$ will not be distributed in such a way to allow $D$ to be written as $D = p_i + c_i$ for all $p_i$.

Proof of theorem 5 is the ultimate goal for the research.