Lyme Disease: A Mathematical Approach

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July 21, 2015

Overview

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Why is Lyme Disease Important to Study?

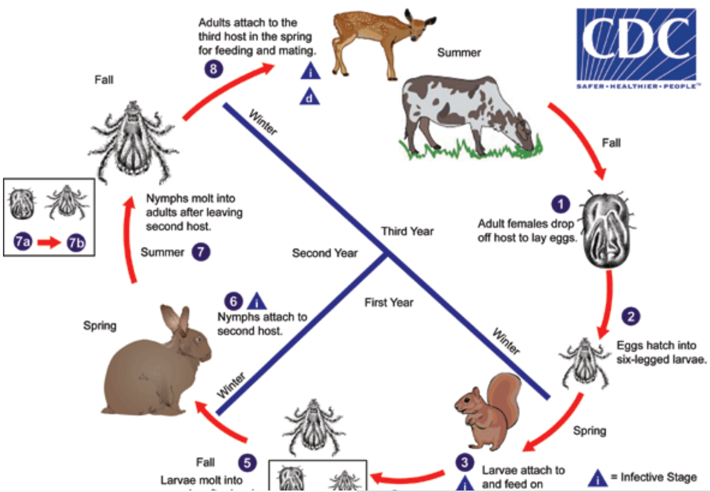
- The shortening of Winter in the North led to the following:
- Warmer temperatures have been predicted to both enhance transmission intensity and extend the distribution of diseases such a malaria and dengue as well.
- Climate change may open up previously uninhabitable territory for arthropod vectors as well as increase reproductive and biting rates, and shorten the pathogen incubation period.

Borrelia Burgdoferi

Ixodes scapularis

- *Ixodes scapularis*, the black-legged tick
- Can be found throughout the country **including Texas**
- Take a blood meal every time they molt
- Once infected, they are infected for life
- Must attach for 36 hours to transmit the bacteria
- No vertical transmission
- Questing season is changing due to climate change

Figure of Life cycle:



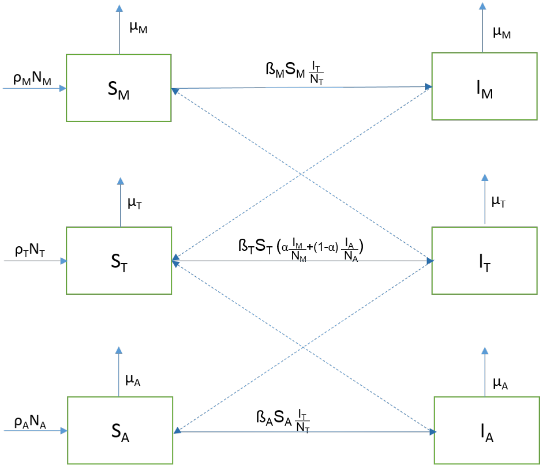
White footed Mouse, Peromyscus leucopus

Unidentified Alternate Host



1. Recent data shows that ticks quest at two distinct heights
2. This will help narrow down the search for the other hosts

Compartmental Model for Base Model



Base Model

$$\frac{di_M}{dt} = \alpha\tilde{\beta}_M(1 - i_M)i_T - \mu_M i_M$$

$$\frac{di_T}{dt} = \tilde{\beta}_T(1 - i_T)\Big((\alpha i_M) + (1 - \alpha)i_A\Big) - \mu_T i_T$$

$$\frac{di_A}{dt} = (1 - \alpha)\tilde{\beta}_A(1 - i_A)i_T - \delta_1\mu_A i_A$$

Base Model with Seasonality

$$\frac{di_M}{dt} = \alpha\tilde{\beta}_M(1 - i_M)i_T - \mu_M i_M$$
$$\frac{di_T}{dt} = \tilde{\beta}_T(1 - i_T)\Big((\alpha i_M) + (1 - \alpha)i_A\Big) - \mu_T i_T$$
$$\frac{di_A}{dt} = \delta_1(1 - \alpha)\tilde{\beta}_A(1 - i_A)i_T - \delta_1\mu_A i_A$$

middle $\frac{2}{3}$:61-304

$$\delta_1 = \begin{cases} 1 & \text{if } 61 \leq t \leq 304 \\ 0 & \text{Otherwise} \end{cases}$$

Parameters

Parameter	Definition
ρ_M	Birth rate of the mice into the susceptible class
ρ_T	Birth rate of the ticks into the susceptible class
ρ_A	Birth rate of the alternate host into the susceptible class
β_M	Contact Transmission Rate for the mice
β_T	Contact Transmission Rate for the Tick
β_A	Contact Transmission Rate for the alternate host
α	Proportion of the the ticks that have a preference for questing at lower heights
μ_M	Death rate for the Mouse class
μ_T	Death rate for the Tick class
μ_A	Death rate for the Alternate Host class

Table: Table of Variables & Parameters for Base Model

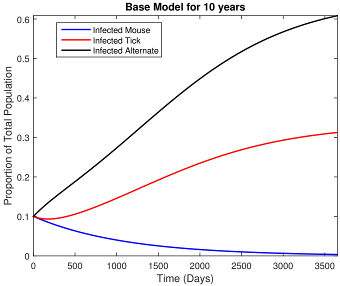
Basic Reproduction Number of a Infection

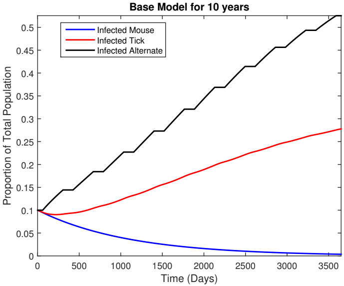
The nondimensionalized system was reduced and rearranged into an equation for $\mathscr{R}_0$, which determines whether or not there will be an epidemic.

- If $\mathscr{R}_0 < 1$ then the disease will eventually die out of the population.
- If $\mathscr{R}_0 = 1$ the disease remains at a constant level in the population.
- If $\mathscr{R}_0 > 1$ the level of disease in the population will increase until there is an epidemic.

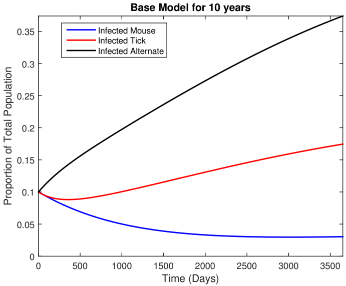
$$\mathscr{R}_0 := \sqrt{\frac{\alpha^2 \beta_M \beta_T \mu_A + (1-\alpha)^2 \beta_A \beta_T \mu_M}{\mu_M \mu_T \mu_A}}$$

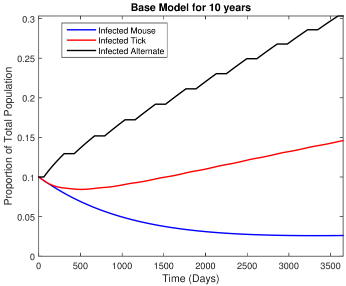
Scenarios in which the overall $\mathscr{R}_0 > 1$ and an epidemic will occur in the community.



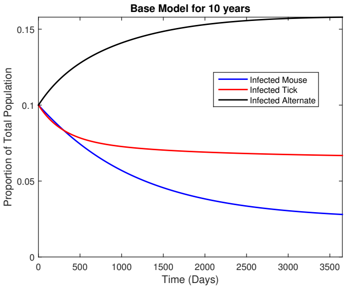


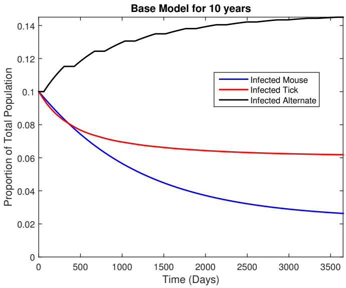
Overall $\mathscr{R}_0 > 1$ and an epidemic occurs



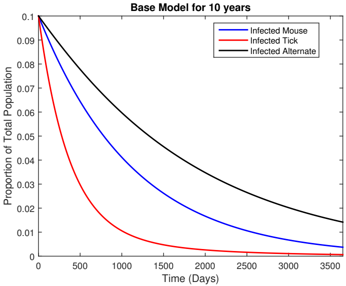


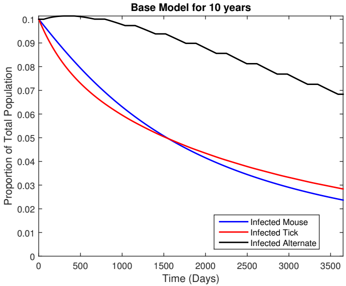
Scenarios in which the overall $\mathscr{R}_0 > 1$ and an epidemic will occur in the community Cont.





Overall $\mathscr{R}_0 < 1$ and the disease dies out





Conclusion

- Adding an invading species with $\mathscr{R}_0 > 1$ can increase the level of infection for the initial host

Three Tick Stages

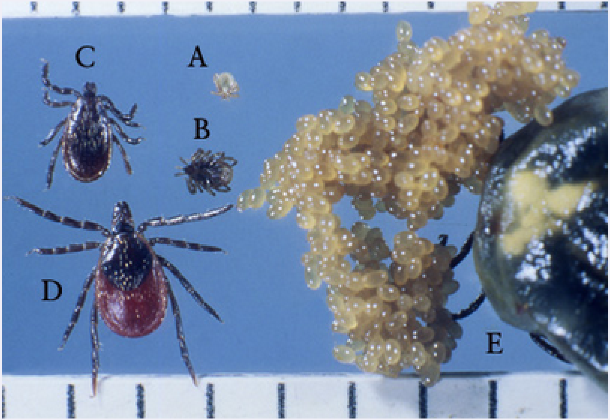
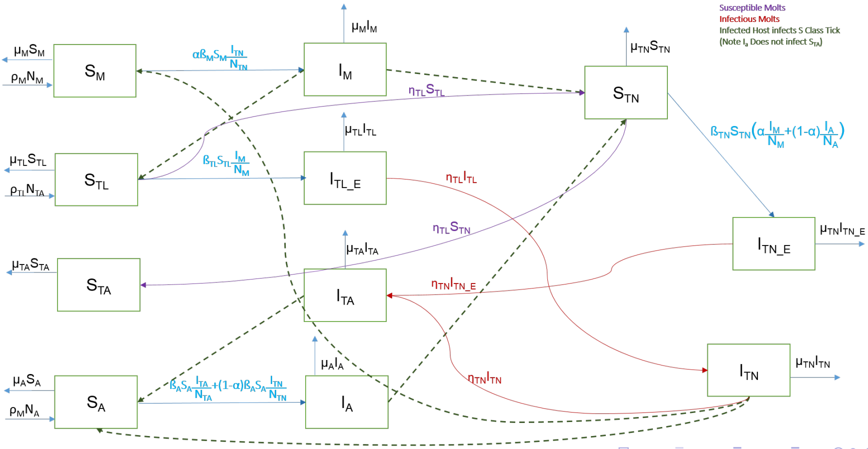


Figure A: Larva (**A**), nymph (**B**), adult male (**C**), adult female (**D**), and engorged adult female with eggs (**E**) of *I. scapularis*. Image courtesy of James Occi.

System of ODEs Modeling the Spread of Lyme Disease

Diagram that incorporates Criss-Cross Infection



System of ODEs Modeling the Spread of Lyme Disease

$$\frac{di_M}{dt} = \alpha\beta_M(1-i_M)i_{T_N} - \mu_M i_M \tag{1a}$$

$$\frac{di_{T_L}}{dt} = \beta_{T_L}(1-i_{T_L})i_M - \eta_{T_L} i_{T_L} - \mu_{T_L} i_{T_L} \tag{1b}$$

$$\frac{di_{T_{N\_E}}}{dt} = \beta_{T_N}(1-i_{T_{N\_E}} - i_{T_N})\Big(\alpha i_M + (1-\alpha)i_A\Big) - \eta_{TN} i_{TN\_E} - \mu_{T_N} i_{TN\_E} \tag{1c}$$

$$\frac{di_{T_N}}{dt} = \eta_{T_L} i_{T_L} - \eta_{T_N} i_{T_N} - \mu_{T_N} i_{T_N} \tag{1d}$$

$$\frac{di_{T_A}}{dt} = \eta_{T_N}(i_{T_N} + i_{TN\_E}) - \mu_{T_A} i_{T_A} \tag{1e}$$

$$\frac{di_A}{dt} = \beta_A(1-i_A)\Big(i_{T_A} + (1-\alpha)i_{T_N}\Big) - \mu_A i_A \tag{1f}$$

Parameters

Parameter	Definition
ρ_M	Birth rate of the mice into the susceptible class
ρ_T	Birth rate of the ticks into the susceptible class
ρ_A	Birth rate of the alternate host into the susceptible Larvae class
β_M	Contact Transmission Rate for the mice
β_{T_L}	Contact Transmission Rate for the larval tick
β_{T_N}	Contact Transmission Rate for the nymphal tick
β_A	Contact Transmission Rate for the alternate host
η_{T_L}	Rate that the larvae molt into nymphs
η_{T_N}	Rate that the larvae nymphs into adults
α	Proportion of the the ticks that have a preference for questing at lower heights
μ_M	Death rate for the Mouse class
μ_T	Death rate for the Tick class
μ_A	Death rate for the Alternate Host class

Table: Table of Variables & Parameters for Model with 3 Tick Classes

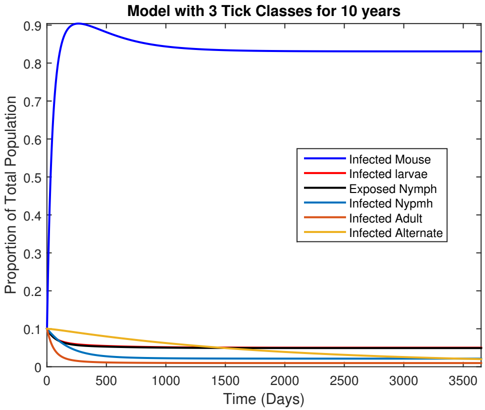
Basic Reproduction Number of a Infection

$$\mathscr{R}_0 := \max\left\{\sqrt{\frac{\alpha\beta_M\beta_L\eta_{T_L}}{\mu_M(\eta_{T_L}+\mu_{T_L})(\eta_{T_N}+\mu_{T_N})}},\quad \sqrt{\frac{\beta_{T_N}(1-\alpha)\beta_A\eta_{T_N}}{(\eta_{T_N}+\mu_{T_N})\mu_{T_A}\mu_A}}\right\}$$

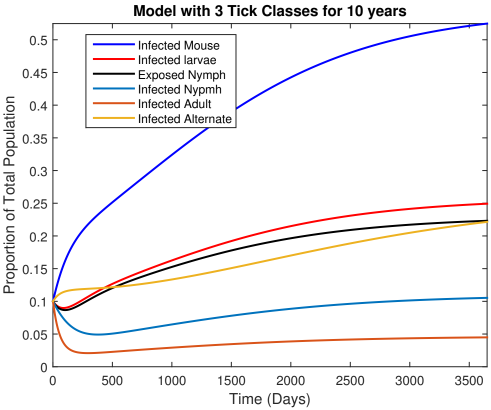
To illustrate that the disease will be endemic in the alternate host. We chose variables in the following manner in the following manner:

- higher β_M
- lower β_{T_L}, β_{T_N}, and β_A

Model with 3 Tick Classes $\alpha = 1, \beta_M = .21, \beta_{T_L} = .00041, \beta_{T_N} = .00041, \beta_A = .00041, \mathscr{R}_0 = 6.2214$



Model with 3 Tick Classes $\alpha = 1, \beta_M = .01, \beta_{T_L} = .0041, \beta_{T_N} = .0041, \beta_A = .0041, \mathscr{R}_0 = 2.9626$



Conclusion

- It's very beneficial for the mice to be able to sustain the disease, but not necessary.
- If the the disease is endemic to the mice population, then it's very likely that the disease will be endemic for the alternate host population.

System of ODEs Modeling the Spread of Lyme Disease

$$\frac{di_M}{dt} = \delta_3 \alpha \beta_M (1 - i_M) i_{T_N} - \mu_M i_M \tag{2a}$$

$$\frac{di_{T_L}}{dt} = \delta_1 \beta_{T_L} (1 - i_{T_L}) i_M - \eta_{T_L} i_{T_L} - \mu_{T_L} i_{T_L} \tag{2b}$$

$$\frac{di_{T_{N\_E}}}{dt} = \delta_3 \beta_{T_N} (1 - i_{T_{N\_E}} - i_{T_N}) \Big(\alpha i_M + (1 - \alpha) i_A \Big) - \eta_{TN} i_{TN\_E} - \mu_{T_N} i_{TN\_E} \tag{2c}$$

$$\frac{di_{T_N}}{dt} = \eta_{T_L} i_{T_L} - \eta_{T_N} i_{T_N} - \mu_{T_N} i_{T_N} \tag{2d}$$

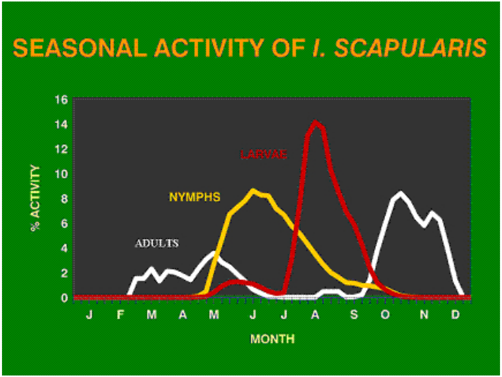
$$\frac{di_{T_A}}{dt} = \eta_{T_N} (i_{T_N} + i_{TN\_E}) - \mu_{T_A} i_{T_A} \tag{2e}$$

$$\frac{di_A}{dt} = \beta_A (1 - i_A) \Big(\delta_5 i_{T_A} + \delta_3 (1 - \alpha) i_{T_N} \Big) - \mu_A i_A \tag{2f}$$

$$\delta_1 = \begin{cases} 1 & \text{if active} > 182 \text{ active} < 283 \\ 0 & \text{not active} \end{cases}$$

$$\delta_3 = \begin{cases} 1 & : \text{active} > 119 \text{ and active} < 283 \\ 0 & \text{not active} \end{cases}$$

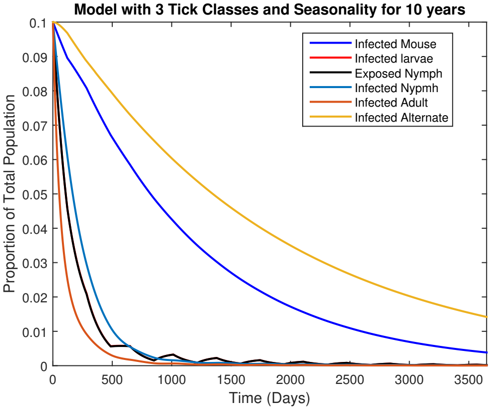
$$\delta_5 = \begin{cases} 1 & \text{active} > 274 \text{ active} < 346 \; or \text{active} > 41 \text{ active} < 161 \\ 0 & \text{not active} \end{cases}$$


SEASONAL ACTIVITY OF I. SCAPULARIS
% ACTIVITY
16
14
12
10
8
6
4
2
0
LARVAE
NYMPHS
ADULTS
J F M A M J J A S O N D
MONTH

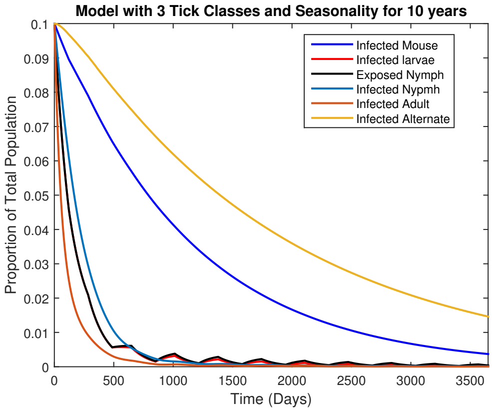
Basic Reproduction Number of a Infection

$$\mathscr{R}_0 := \begin{cases} 0 & 0 \leq t \leq 121 \\ \sqrt{\dfrac{\alpha\beta_M\beta_{T_L}\eta_{T_L}}{(\eta_{T_L}+\mu_{T_L})(\eta_{T_N}+\eta_{T_N})\mu_M}} & 121 \leq t \leq 274 \\ max\left\{\sqrt{\dfrac{\alpha\beta_M\beta_L\eta_{T_L}}{\mu_M(\eta_{T_L}+\mu_{T_L})(\eta_{T_N}+\mu_{T_N})}}, \quad \sqrt{\dfrac{\beta_{T_N}(1-\alpha)\beta_A\eta_{T_N}}{(\eta_{T_N}+\mu_{T_N})\mu_{T_A}\mu_A}}\right\} & 274 \leq t \leq 283 \\ 0 & 283 \leq t \leq 346 \\ 0 & 346 \leq t \leq 365 \end{cases}$$

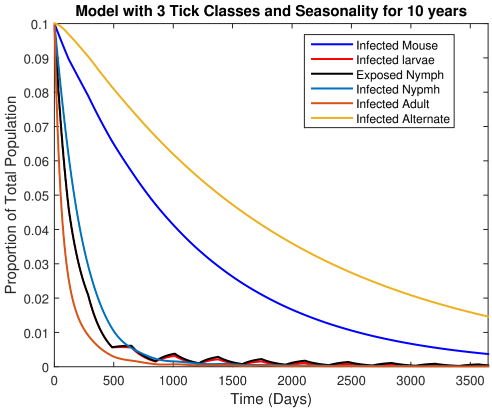
Model with Seasonality: $\alpha = 1, \beta_M = .0011, \beta_{T_L} = .0011, \beta_{T_N} = .0011, \beta_A = .0011, \mathscr{R}_0 = .8693$



Model with Seasonality: $\alpha = .5, \beta_M = .011, \beta_{T_L} = .011, \beta_{T_N} = .011, \beta_A = .011, \mathscr{R}_0 = .7561$



Model with Seasonality: $\alpha = 0, \beta_M = .0011, \beta_{T_L} = .0011, \beta_{T_N} = .0011, \beta_A = .0011, \mathscr{R}_0 = .7036$



Conclusion

- Lyme disease is found in mice in Texas at low levels. The model indicates that it's very likely that there is a larger host that is also an effective carrier.

References

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