Gröbner Bases and the Neural Ideal

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Place cells are neurons that encode spatial information.

We can describe the activity of $n$ with binary strings of length $n$.

- 1 denotes a firing neuron
- 0 denotes a silent neuron
A neural code

Definition
Given a set of neurons labeled \( \{1, \ldots, n\} \), a neural code on \( n \) neurons is a set of binary strings \( C \subset \{0, 1\}^n \).

Example
Let us consider the following code on 3 neurons:
\[ C = \{100, 110, 010, 011, 001\} \]
Ideals and Varieties

Definition

**Ideals:** Let $R$ be a commutative ring. A subset $I \subset R$ is an **ideal** of $R$ if it has the following properties:

1. $I$ is a subgroup of $R$ under addition.
2. If $a \in I$, then $ra \in I$ for all $r \in R$.

An ideal $I$ is said to be **generated** by a set $A$, and we write $I = \langle A \rangle$, if $I$ is the set of all finite combinations of elements of $A$ with coefficients in $R$.

Definition

Let $J \subset \mathbb{F}_2[x_1, \ldots, x_n]$ be an ideal, and define the variety

$$V(J) = \{ v \in \{0, 1\}^n \mid f(v) = 0 \text{ for all } f \in J \}.$$
Definition

For some $f \in \mathbb{F}_2[x_1, \ldots, x_n]$, $f$ is a pseudo-monomial if $f$ has the form

$$f = \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 + x_j) = x_\sigma \prod_{i \in \tau} (1 + x_i).$$

for some $\sigma, \tau \subset [n]$ with $\sigma \cap \tau = \emptyset$.

Example

Pseudo-monomial:

$$x_1x_2(1 + x_3)(1 + x_4)$$

Not pseudo-monomials:

$$x_1x_2 + x_1x_3 \text{ and } x_1^2x_2$$
The neural ideal

Definition
For any $v \in \{0, 1\}^n$, consider $\rho_v$, defined as

$$\rho_v = \prod_{i=1}^{n} (1 - v_i - x_i) = \prod_{\{i | v_i = 1\}} x_i \prod_{\{j | v_j = 0\}} (1 + x_j)$$

Definition
For a code $C$, the neural ideal $J_C = \langle \{\rho_v | v \notin C\} \rangle$.
Note: $V(J_C) = C$.

Definition
The canonical form of a neural ideal $J_C$ is the set of all minimal pseudo-monomials that are elements of $J_C$. 
The canonical form gives us a compact description of the relationships between receptive fields associated with a code.

**The ultimate goal:**
To find an efficient method to compute the canonical form of a neural code.

- Computing the canonical form is very computationally inefficient
- The computation is infeasible for codes on large numbers of neurons.
- (Petersen et al) Computing the Gröbner basis, another generating set of the ideal, is much faster.
Gröbner bases

Definition
A set $\{g_1, \ldots, g_t\} \subseteq I$ is a Gröbner basis of $I$ if and only if the leading term of any element of $I$ is divisible by one of the $\text{LT}(g_i)$.

Definition (Criteria for a reduced Gröbner basis)
Let $G$ be a Gröbner basis. $G$ is a reduced Gröbner basis for all $g \in G$, no trailing term of any $g \in G$ is divisible by the leading term of any element of $G$.
Note: For a given monomial order the reduced Gröbner basis is unique.

Definition
Let $I$ be an ideal. The universal Gröbner basis is the union of all the reduced Gröbner bases of $I$ w.r.t. any monomial order.
Theorem (L.)

Let $f$ be an pseudo-monomial, and let $G = \{g_1, \ldots, g_k\}$ be a set of pseudo-monomials. If the remainder on division of $f$ by $G = \{g_1, \ldots, g_s\}$ is 0 for any monomial ordering, then for some $g \in G$, $g$ divides $f$. 
Proposition

Let \( f = x_\sigma \prod_{i \in \tau} (1 + x_i) \) be a pseudo-monomial. Then we can write \( f \) as

\[
    f = \sum_{\gamma \in P(\tau)} x_\sigma x_\gamma,
\]

where \( P(\tau) \) is the powerset of \( \tau \).

Notice that each term of \( f \) corresponds to an element of \( P(\tau) \).
Example

Let $f = x_1(1 + x_2)(1 + x_3)(1 + x_4)$. In this case, $\sigma = \{1\}$ and $\tau = \{2, 3, 4\}$.

Hypercube of $f$
Lemma

Let $J \in \mathbb{F}_2[x_1, \ldots, x_n]$ be an ideal, and let $f, g$ be pseudo-monomials such that $g = x^\alpha \prod_{i \in \beta} (1 + x_i)$ and $f = x^\sigma \prod_{j \in \tau} (1 + x_j)$. Then $g \mid f$ if and only if $\alpha \subset \sigma$ and $\beta \subset \tau$.

Lemma

Let $f = x^\sigma \prod_{i \in \tau} (1 + x_i)$ and let $H$ be the hypercube of $P(\sigma \cup \tau)$. A pseudo-monomial $h$ divides $f$ if and only if the hypercube of $h$ is a sub-cube of $H$ and the hypercube of $h$ intersects the Hasse diagram of $P(\sigma)$ at a unique vertex.
Geometric Intuition
Proof Idea

Initial State

End state
Theorem (L.)

Let $C$ be a code, and $J_C$ be the neural ideal of $C$. If the canonical form of $J_C$ is a Gröbner basis, then the canonical form of $J_C$ is a reduced Gröbner basis.

Theorem (L.)

Let $J_C$ be a neural ideal and let $G$ be the universal Gröbner basis of $J_C$. For all $g \in G$, if $g$ is a pseudo-monomial, then $g$ is in the canonical form of $J_C$. 
Application

1. Compute Gröbner basis of neural ideal
2. If all $g \in GB$ are pseudo-monomials, return GB
3. Improves runtime when $CF = GB$
Complementary Codes

Definition

Let \( c \in \{0, 1\}^n \) be a code. The **complement** of \( c \) is the code \( c' \in \{0, 1\}^n \) such that \( c'_i = 1 \) if and only if \( c_i = 0 \).

Definition

Let \( f \) be a pseudo-monomial such that \( f = x_\sigma \prod_{i \in \tau} (1 + x_i) \). The **complement** of \( f \), denoted \( f' \), is \( f' = x_\tau \prod_{j \in \sigma} (1 + x_j) \).

Definition

A code \( C \subset \{0, 1\}^n \) is called **complement-complete** if for all \( c \in C \), \( c' \in C \) as well.
Theorem (L.)

Let $C$ be a code on $n$ neurons such that $C \subseteq \{0, 1\}^n$. If $C$ is complement-complete, then the canonical form of $J_C$ is not a Gröbner basis.

Example

Let $C = \{111, 000, 110, 001, 100, 011\}$.  
$C' = \{010, 101\}$

$J_C = \langle x_2(1 + x_1)(1 + x_3), x_1x_3(1 + x_2) \rangle$
Future work

What we know now: If an element of the Gröbner basis is a pseudo-monomial, then it is in the canonical form.

What we want to know: If an element of the Gröbner basis is not a pseudo-monomial, can we still use it to find elements of the canonical form?
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