Counting the $p$-adic valuations of the roots of multivariate systems of polynomials

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Paths of Glory before ours

- 1637: Descarte’s Rule. Suppose $f \in \mathbb{R}[x_1]$ and has $t$ terms. Then there are at most $2t + 1$ real roots.

- 1980s: van den Dries and (?). Suppose $f_i, \ldots, f_n \in \mathbb{Q}[x_1, \ldots, x_n]$ with $\leq t$ terms each. Then there exists a finite number of isolated roots in $\mathbb{Q}_p^n$. No explicit formula found yet!

- 2000s: $p$-adic tropical geometry can help with finding explicit bounds on the number of roots in $\mathbb{Q}_p^n$. Complexity theory gets involved!

Our goal this summer

Use $p$-adic techniques to help bound the number of integers roots of certain polynomial systems.
Let \( p \) be prime.

- \( \mathbb{Z} \): Field of integers. An integer in base 3 is a finite sequence. 
  Ex: \( 1012 \) (base 3) = \( 1 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0 \) (base 10)

- \( \mathbb{Z}_3 \): All sequences terminating on the right. 
  \( \cdots 1012 = \cdots + 1 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0 \)

- \( \mathbb{Q}_3 \): All sequences with a finite number of digits after the decimal point. 
  \( \cdots 1012.22 = \cdots + 1 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0 + 2 \cdot 3^{-1} + 2 \cdot 3^{-2} \)

- \( \mathbb{C}_3 \): The completion of the algebraic closure of \( \mathbb{Q}_3 \).
Motivating example: 3-adic roots of $162 - x + 63x^3$

Let $p = 3$. Consider the polynomial $162 - x + 63x^3$. One real root ($\approx -1.373\ldots$), but three 3-adic roots. Found in Maple:

\[
3^{-1} + 2 + 2 \cdot 3 + 2 \cdot 3^4 + 2 \cdot 3^5 + \text{higher order terms}
\]
\[
2 \cdot 3^{-1} + 2 \cdot 3^2 + 2 \cdot 3^3 + 3^4 + \text{higher order terms}
\]
\[
2 \cdot 3^4 + 2 \cdot 3^{14} + 3^{15} + \text{higher order terms}
\]

The power of 3 of the first non-zero term is its $p$-adic valuation.

**Upshot**

The polynomial $162 - x + 63x^3$ has three 3-adic roots. Two roots have valuation $-1$ and one root has valuation 4.
Drawing pictures

You could also draw a picture to get to the same upshot.

\[ f(x) = 162 - x + 63x^3 = 2 \cdot 3^4 - x + 7 \cdot 3^2 \]

For \( f \in \mathbb{C}_p[x_1, \ldots, x_n] \) written \( f = \sum_i c_i x^{a_i} \):

**Definition (Newton polytope of \( f \))**

The *Newton polytope*, \( \text{Newt}(f) \), is the convex hull of the set \( \{a_i\} \).

**Definition (\( p \)-adic Newton polytope of \( f \))**

The *\( p \)-adic Newton polytope*, \( \text{Newt}_p(f) \), is the convex hull of the set \( \{a_i, \text{ord}_p(c_i)\} \).

**Definition (\( p \)-adic Tropical Variety of \( f \))**

The *\( p \)-adic Tropical Variety*, \( \text{Trop}_p(f) \), is the set \( \{v \in \mathbb{R}^n | (v, 1) \text{ is an inner normal of a positive-dim. face of } \text{Newt}_p(f)\} \).
Let $p = 3$. Consider the polynomial $g = 1 + x^2 - 54xy$.

What does $\text{Trop}_p(g)$ look like? (Demonstration)

**Upshot**

We can derive the $Y$-shape of $\text{Trop}_p(g)$ by $\text{Newt}(g)$ (independent of coefficients).

If you want the position of $\text{Trop}_p(g)$, you need $\text{Newt}_p(g)$ (dependent of coefficients).

Cory-jargon: The $Y$-shape in $\text{Trop}_p(f_i)$ will occur if $\text{Newt}(f_i)$ is a triangle. They are “hyper-$Y$’s.”
Theorem (Kapranov)

For a system $F := (f_1, \cdots, f_n) \in \mathbb{C}_p[x_1, \cdots, x_n]$, 

$$\text{ord}_p \left( \mathbb{Z}_{\mathbb{C}_p}(f_1, \cdots, f_n) \right) \subseteq \bigcap_{i=1}^n \text{Trop}_p(f_i) \cap \mathbb{Q}^n.$$ 

Let $t :=$ the number of exponent vectors, $\{a_i\}$ in the system.

Special case (the “circuit case”): When $t = n + 2$ (with some mild conditions). Use Gaussian Elimination and reduce problem to looking at a collection of hyper-Y’s.

Goal

Find a sufficiently good upper bound on the number of intersections of the $\text{Trop}_p(f_i)$ in the case where $t = n + 2$. 
Higher dimensional hyper-Y’s and why we choose $p$-adics

$$F := (f_1, f_2, f_3) := (xy - x^2 - 1/16^6, yz - 1 - x^2, z - 1 - x^2/16^{18})$$
We have $t = n + 2$. $xy, x^2, 1, yz, z$

Look at intersections of $\text{ArchTrop}(f_1) \cap \text{ArchTrop}(f_2) \cap \text{ArchTrop}(f_3)$. (Image: Dr. Rojas.)

$\text{ArchTrop}(f_i)$ is the real analog to $\text{Trop}_p(f_i)$. 

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Old Bounds and New

Theorem (Koiran, Portier, Rojas)

Suppose $F := (f_1, \ldots, f_n)$ with $f_i \in \mathbb{C}_p(x_1, \ldots, x_n)$. In the “circuit case” ($\#$ exponent vectors $= n + 2$), then the maximum number of valuations of the roots of $F$ is at most $\max\{2, \left\lfloor \frac{n}{2} \right\rfloor^n + n\}$.

Short-term goal

Achieve an upper bound polynomial in $n$. For certain case nice cases, we can prove a bound of $n + 1$. For certain less-nice cases, a bound of $2n + 1$ (S).

Conjecture (Koiran, Portier, Rojas)

The bound can be improved to $n + 1$. This bound is sharp.
Conclusions

Same goal, new friend

Find sufficiently good bounds on the number of integer roots for a system of multivariate polynomials.

Bounding $p$-adic valuations is a step towards bounding integer roots. We do this by looking at intersections of the $\text{Trop}_p(f_i)$'s.

In the “circuit case,” we want to bound the number of intersections ($=$ upper bound on number of valuations of the roots of $F$) by $n + 1$. 
Thank you

Thank you for listening!
References


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