Nonvanishing of Hecke L-Series and $\ell$-torsion in Class Groups

Arianna Iannuzzi, Alex Mathers, and Maria Ross

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Introduction

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Group Characters

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- The set of characters of $G$ form a group.
- A Dirichlet character of modulus $m$ is a group character for $G = (\mathbb{Z}/m\mathbb{Z})^*$, or equivalently a multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

\[
\begin{align*}
(i) \quad & \chi(n + m) = \chi(n) \text{ for all } n, \\
(ii) \quad & \chi(n) = 0 \text{ for } \gcd(n, m) > 1.
\end{align*}
\]
If $\chi$ is a Dirichlet character, then the $L$-series of $\chi$ is defined by the series

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{Re}(s) > 1.$$
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Example: The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1.$$
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Functional Equation

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Functional Equation

- The $L$-series of a Dirichlet character $\chi$ extends to a meromorphic function on the entire complex plane via *analytic continuation*.
- This analytic continuation satisfies a *functional equation* of the form $s \mapsto 1 - s$ with central value $L(\chi, 1/2)$.
- Example: If we let $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$, then we have the functional equation

\[ \xi(s) = \xi(1 - s) \]

and the central value is given by $\zeta(1/2)$. 
Our “set up”

- Fix a triple of integers \((d, k, D)\) satisfying:
  - \(d \equiv 1 \pmod{4}\),
  - \(k > 0, \quad \text{sign}(d) = (-1)^{k-1}\),
  - \(D > 0, \quad D \equiv 7 \pmod{8}, \quad \gcd(d, D) = 1\).
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- Let \(K\) be the imaginary quadratic field \(K = \mathbb{Q}(\sqrt{-D})\).
The Class Group

- If \( \mathcal{O}_K \) denotes the ring of integers of \( K \), then \( K \) can be considered as an \( \mathcal{O}_K \)-module. Denote the set of fractional ideals of \( \mathcal{O}_K \) by \( I_K \).
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- The set \( I_K \) is an abelian group under multiplication; denote the subgroup of “principal” ideals by \( P_K \), and set

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\text{Cl}(K) = I_K / P_K.
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The Class Group

- If $\mathcal{O}_K$ denotes the ring of integers of $K$, then $K$ can be considered as an $\mathcal{O}_K$-module. Denote the set of fractional ideals of $\mathcal{O}_K$ by $I_K$.

- The set $I_K$ is an abelian group under multiplication; denote the subgroup of “principal” ideals by $P_K$, and set

$$\text{Cl}(K) = I_K/P_K.$$ 

- This is called the class group. It is finite, and its order (the class number) is denoted $h(-D)$. 
A canonical Hecke character for some “distinguished subgroup” $I_D$ of $I_K$ is, roughly speaking, a character $\psi_k : I_D \to \mathbb{C}^*$ which can be decomposed into a “finite part” and “infinite part”, and satisfies

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A canonical Hecke character for some “distinguished subgroup” $I_D$ of $I_K$ is, roughly speaking, a character $\psi_k : I_D \to \mathbb{C}^*$ which can be decomposed into a “finite part” and “infinite part”, and satisfies

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Given such a $\psi_k$, we can define its “quadratic twist” $\psi_{d,k}$. We denote the set of all $\psi_{d,k}$ by $\Psi_{d,k}(D)$; there are exactly $h(-D)$ such characters.
Hecke $L$-series

- To a canonical Hecke character $\psi \in \Psi_{d,k}(D)$, we can assign an $L$-series $L(\psi, s)$, which converges for $\text{Re}(s) > k + \frac{1}{2}$. 

This Hecke $L$-series has an analytic continuation satisfying $s \mapsto 2k - s$, $L(\psi, s) = L(\psi, 2k - s)$. 

We are interested in the central value $L(\psi, k)$, specifically in determining whether it is zero or nonzero.
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- We are interested in the central value $L(\psi, k)$, specifically in determining whether it is zero or nonzero.
Arithmetic Significance

Let $d = k = 1$. Then our characters $\psi \in \Psi_{1,1}(D)$ naturally correspond to canonical examples of Gross’s $\mathbb{Q}$-curves over $K = \mathbb{Q}(\sqrt{-D})$. If $A(D)$ is such an elliptic curve, then

$$L(A(D), s) = \prod_{\psi \in \Psi_{1,1}(D)} L(\psi, s).$$
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If $L(\psi, 1) \neq 0$ for some $\psi \in \Psi_{1,1}(D)$, then $L(\psi, 1) \neq 0$ for all $\psi \in \Psi_{1,1}(D)$, hence $L(A(D), 1) \neq 0$. 

By known results towards the BSD conjecture, this implies that the rank of $A(D)$ is zero, and hence the group of $K$-rational points is finite.
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Statement of Results

Arianna Iannuzzi

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Outlining our Goals

Since \( \# \Psi_{d,k}(D) = h(-D) \), by Siegel’s theorem

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h(-D) \gg \epsilon \ D^{\frac{1}{2} - \epsilon}
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  we have
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  \#\Psi_{d,k}(D) \gg \epsilon D^{\frac{1}{2}-\epsilon}.
  \]

- We would like to quantify the number of \( \psi \in \Psi_{d,k}(D) \) with nonvanishing central value. Therefore we define
  \[
  NV_{d,k}(D) = \#\{\psi \in \Psi_{d,k}(D) : L(\psi, k) \neq 0\}.
  \]
Outlining our Goals

- We would like to find a bound of the form

\[ NV_{d,k}(D) \gg D^{\delta_k} \]

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Outlining our Goals

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$$NV_{d,k}(D) \gg D^{\delta_k}$$

for some $\delta_k > 0$.

- Previous results of this form holding for all values of $D$ have been conditional on the GRH.

- Our work has involved eliminating the GRH hypothesis. Doing so, we can no longer guarantee that our bound will hold for all values of $D$, but we can guarantee that it will be true “100 percent of the time”!
Definitions

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- Let $\mathcal{S}_{d,k}(X)$ be the subset of $\mathcal{S}_{d,k}$ such that $D \leq X$. 
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- Let $S_{d,k}$ be the set of all imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ satisfying our conditions on $(d, k, D)$, plus some additional “local conditions”.

- Let $S_{d,k}(X)$ be the subset of $S_{d,k}$ such that $D \leq X$.

- Let $S^{NV}_{d,k}(X)$ be the subset of $S_{d,k}(X)$ satisfying the bound

$$NV_{d,k}(D) \gg \epsilon \frac{1}{D^{2(2k-1)} - \epsilon}.$$
Main Results

Theorem

We have the asymptotic formula

$$\#S_{d,k}^N(X) = \delta_{d,k}X + O_{d,k}(X^{1 - \frac{1}{2(2k-1)}})$$

as $X \to \infty$, for some explicit positive constant $\delta_{d,k}$. 
Main Results

Theorem

We have the asymptotic formula

\[
\frac{\#S_{d,k}^{NV}(X)}{\#S_{d,k}(X)} = 1 + O(X^{-\frac{1}{2(2k-1)}})
\]

as \(X \to \infty\). In particular, the bound

\[
NV_{d,k}(D) \gg \epsilon D^{\frac{1}{2(2k-1)}} - \epsilon
\]

holds for 100% of imaginary quadratic fields \(K \in S_{d,k}\).
Outline of Proof

Maria Ross

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Galois Orbit

We define the Galois group $G_k = \text{Gal}(\overline{\mathbb{Q}}/K(\zeta_{2k-1}))$, where $\zeta_{2k-1}$ denotes a primitive $2k - 1^{st}$ root of unity.

Then $G_k$ acts on the set of characters $\Psi_{d,k}(D)$ by

$$\psi \mapsto \psi^\sigma,$$

where $\psi^\sigma = \sigma \circ \psi$ for $\sigma \in G_k$, and the Galois orbit of a character $\psi$ is

$$O_{\psi} = \{\psi^\sigma : \sigma \in G_k\}.$$
Strategy of Proof

- First, we find one character $\psi$ so $L(\psi, k) \neq 0$. We do so by proving the following theorem:
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**Theorem**

If $D > 64d^4(k + 1)^4$, there exists a $\psi \in \Psi_{d,k}(D)$ such that $L(\psi, k) \neq 0$. 


Strategy of Proof

Then, we use results of Shimura to show that

\[ L(\psi, k) \neq 0 \iff L(\psi^\sigma, k) \neq 0 \]

for all \( \sigma \in G_k \).
Strategy of Proof

- Then, we use results of Shimura to show that

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for all \( \sigma \in G_k \).

- It follows that \( NV_{d,k}(D) \geq \#O_{\psi} \).
Strategy of Proof

Let $\text{Cl}_\ell(K)$ be the $\ell$—torsion subgroup of the class group $\text{Cl}(K)$. 
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- By Rohrlich, we have that under certain “local conditions”,

$$\#\mathcal{O}_\psi = \frac{h(-D)}{|\text{Cl}_{2k-1}(K)|}.$$
Strategy of Proof

Let $\text{Cl}_\ell(K)$ be the $\ell$–torsion subgroup of the class group $\text{Cl}(K)$.

- By Rohrlich, we have that under certain “local conditions”,
  \[
  \# \mathcal{O}_\psi = \frac{h(-D)}{|\text{Cl}_{2k-1}(K)|}.
  \]

- Now we want to find a lower bound of the form
  \[
  \frac{h(-D)}{|\text{Cl}_{2k-1}(K)|} \gg D^{\delta_k}
  \]
  for some $\delta_k > 0$. 
Strategy of Proof

Recall that Siegel’s Theorem gives us the bound

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Recall that Siegel’s Theorem gives us the bound

\[ h(-D) \gg \epsilon D^{\frac{1}{2} - \epsilon}. \]

We want to find an upper bound of the form

\[ |\text{Cl}_{2k-1}(K)| \ll D^{\frac{1}{2} - \delta_k + \epsilon}. \]
Strategy of Proof

Recall that Siegel’s Theorem gives us the bound

\[ h(-D) \gg \epsilon \ D^{1/2 - \epsilon}. \]

We want to find an upper bound of the form

\[ |\text{Cl}_{2k-1}(K)| \ll D^{1/2 - \delta_k + \epsilon}. \]

Combining such a bound with Siegel’s theorem would give

\[ NV_{d,k}(D) \geq \#\mathcal{O}_\psi \gg D^{\delta_k - \epsilon}. \]
Bounding $\ell$-torsion in Class Groups

Theorem (Ellenberg and Venkatesh, 2005)

Assuming GRH,

$$|Cl_\ell(K)| \ll_\epsilon D^{\frac{1}{2} - \frac{1}{2\ell} + \epsilon}.$$
Theorem (Ellenberg, Pierce, Wood (2016))

The bound

$$|Cl_\ell(K)| \ll_{\epsilon} D^{\frac{1}{2} - \frac{1}{2\ell}} + \epsilon$$

holds unconditionally for all imaginary quadratic fields $K$ with $D \leq X$ except an “exceptional set” of size $O(X^{1 - \frac{1}{2\ell}})$. 
Bounding the $\ell$-torsion subgroup

A restatement of the results of Ellenberg, Pierce, and Wood (2016) yields

$$\frac{\#\{K : D \leq X, |\text{Cl}_\ell(K)| \ll_\epsilon D^{\frac{1}{2} - \frac{1}{2\ell} + \epsilon}\}}{\#\{K : D \leq X\}} = 1 + O(X^{-\frac{1}{2\ell}}).$$
Under our particular conditions...

Recall that $\mathcal{S}_{d,k}$ is the set of all imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ that satisfy our conditions on $(d, k, D)$, along with some “local conditions”.

We incorporate our local conditions into the work of Ellenberg, Pierce, and Wood to get an asymptotic formula for the number of imaginary quadratic fields $K$ with $D \leq X$ that satisfy our conditions:

$$\#\mathcal{S}_{d,k}(X) = \delta_{d,k}X + O(X^{\frac{1}{2}})$$

for an explicit constant $\delta_{d,k}$. 
Let $S_{d,k}^{Tor}$ denote the subset of $S_{d,k}$ such that the torsion bound is satisfied, i.e., $|\text{Cl}_\ell(K)| \ll \epsilon D^{\frac{1}{2} - \frac{1}{2\ell} + \epsilon}$.

We prove that if $K$ is in the set $S_{d,k}^{Tor}$, then

$$NV_{d,k}(D) \geq \#\mathcal{O}_\psi \gg D^{\frac{1}{2(2k-1)} - \epsilon}.$$ 

Thus, $S_{d,k}^{Tor}$ is a subset of $S_{d,k}^{NV}$, the set of fields in $S_{d,k}$ with

$$NV_{d,k}(D) \gg \epsilon D^{\frac{1}{2(2k-1)} - \epsilon}.$$
Finding an Asymptotic Formula

We can decompose $S_{d,k}(X)$ into the disjoint union of $S_{d,k}^{NV}(X)$ and its complement, $S_{d,k}^{-}(X)$. Then,

$$\#S_{d,k}^{NV}(X) = \#S_{d,k}(X) - \#S_{d,k}^{-}(X).$$

From Ellenberg, Pierce, and Wood, we know that the number of fields with our particular conditions not satisfying the torsion bound is bounded above by $O(X^{1 - \frac{1}{2(2k-1)}})$.

So, we can use $O(X^{1 - \frac{1}{2(2k-1)}})$ as an upper bound for $\#S_{d,k}^{-}(X)$. 
Finding an Asymptotic Formula

Then, we combine our asymptotic formula for $\#S_{d,k}(X)$ with this upper bound on the number of fields that don’t satisfy $NV_{d,k}(D) \gg \epsilon D^{\frac{1}{2(2k-1)}}$ to get

$$\#S_{d,k}^{NV}(X) = \delta_{d,k} X + O_{d,k}(X^{1-\frac{1}{2(2k-1)}})$$

for explicit positive constant $\delta_{d,k}$.

Finally, we consider the ratio of $\#S_{d,k}^{NV}(X)$ to $\#S_{d,k}(X)$ and arrive at our density statement.
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