Zeros of Eisenstein Series
Arising from Dirichlet Characters

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July 18, 2017
Acknowledgements

- Dr. Young
- Victoria Jakicic
- Texas A&M Mathematics Dept.
- National Science Foundation
The special linear group of degree 2 with coefficients in \( \mathbb{Z} \), denoted \( \text{SL}_2(\mathbb{Z}) \) is defined as

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\text{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}
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An important subgroup is the **Hecke congruence subgroup of level \( N \)** of \( \text{SL}_2(\mathbb{Z}) \), defined as

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\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.
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The **special linear group** of degree 2 with coefficients in $\mathbb{Z}$, denoted $\text{SL}_2(\mathbb{Z})$ is defined as

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Let $\mathcal{H}$ denote the upper half-plane

$$\mathcal{H} = \{ x + iy \in \mathbb{C} : y > 0 \}.$$
We can see that $\text{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by the following:

$$\gamma(z) := \frac{az + b}{cz + d}.$$
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The **fundamental domain** $\mathcal{F} = SL_2(\mathbb{Z}) \backslash \mathcal{H}$ is shown as
A map $f : \mathcal{H} \to \mathbb{C}$ is called a **modular form** of weight $k$ if
1) $f$ is holomorphic on $\mathcal{H}$
2) $\lim_{\text{Im}(z) \to \infty} f(z)$ exists
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Weakly modular of weight $k$ means that

$$f(\gamma(z)) = (cz + d)^k f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. 
We define the **slash operator of weight** \( k \) to be

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 f \bigg|_{\gamma} = (cz + d)^{-k} f(\gamma(z))
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$f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight $k$ with respect to $\Gamma_0(N)$ if we replace our last two conditions with:

2) $f$ is weakly modular of weight $k$ with respect to $\Gamma_0(N)$

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The space of these modular forms is denoted $\mathcal{M}_k(\Gamma_0(N))$. 
An Example

The classical **Eisenstein series** is given as

\[ E_k = \frac{1}{2} \sum_{\gcd(c,d)=1} \frac{1}{(cz + d)^k} \]

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**Our Problem (Part 1)**

Where do Eisenstein series on \(\Gamma_0(N)\) vanish?
A modular form has a Fourier expansion given as

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Cusp Forms and Equidistribution

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**Our Problem (Part 2)**

What structure do our zeros display?
Recall From Last Time...

A **Dirichlet character** modulo $n$ is a map $\chi : \mathbb{Z} \to \mathbb{C}$ which is

- totally multiplicative, that is, $\chi(1) = 1$ and $\chi(mn) = \chi(m)\chi(n)$ for all integers $m, n$
- periodic modulo $n$
- identically zero for all integers not coprime to $n$. 

We say that $\chi$ is a **primitive character** modulo $n$ if it is not induced by a character of smaller modulus $k$. 

Given a primitive Dirichlet character $\chi_1$ modulo $q_1$, and a primitive Dirichlet character $\chi_2$ modulo $q_2$, we have the associated Eisenstein series of weight $k$:

$$E_{\chi_1, \chi_2, k}(z) = \sum_{(c,d) = 1} \chi_1(c)\chi_2(d)\left(cq_2z + d\right)^{-k} \in \mathcal{M}_k(\Gamma_0(q_1q_2))$$
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Where is this thing zero?
The \( cz + d \) expansion is "good" for \( \text{Im}(z) \ll \sqrt{k} \) and the Fourier expansion is "good" for \( \text{Im}(z) \gg \sqrt{k} \).

Within a certain horizontal strip, \( E_{\chi_1, \chi_2, k}(z) \) is dominated by just a few terms.
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We assume $q_2$ is prime, and let $a$ be an integer such that $a$ and $a + 1$ are coprime to $q_2$. 
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Figure: For $k = 10$, $q_1 = 3$, $q_2 = 5$
Letting $\theta$ denote the angle of $z$ from the point $a^{q_2^2}$, we have proved that the zeros become distributed evenly with respect to $\theta$ as $k \to \infty$. 

\[ y = \frac{1+\eta}{2\sqrt{3q_2}} \]
Letting $\theta$ denote the angle of $z$ from the point $\frac{a}{q_2}$, we have proved that the zeros become distributed evenly with respect to $\theta$ as $k \to \infty$. 
Let $z = x + iy$, so in a small strip around
\[ \frac{a+1/2}{q_2} - \frac{\epsilon}{q_2 k} \leq x \leq \frac{a+1/2}{q_2} + \frac{\epsilon}{q_2 k}, \]
we have that the main terms are
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g_a(z) := \frac{\chi(-a)}{(q_2 z - a)^k} + \frac{\chi(-a - 1)}{(q_2 z - a - 1)^k}.
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Distribution With Respect to $\theta$

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In this strip, we have that $g_a(z) = 0$ exactly when $z = \frac{a}{q_2} + Re^{i\theta}$ satisfies

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e^{2i\theta k} + (-1)^k \chi_2(a)\chi_2(a+1) = 0.
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\]

If $\theta$ satisfies the above equation, so does $\theta + \frac{n\pi}{k}$ for any $n \in \mathbb{N}$. 
The $W_n$'s

We look in small regions around each zero of $g(a)(z)$.
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**Lemma**

The error term $|E_{\chi_1, \chi_2, k}(z) - g_a(z)|$ vanishes quickly as $k \to \infty$. 
Leading Up to a Theorem...

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Lemma

The error term $|E_{\chi_1, \chi_2, k}(z) - g_a(z)|$ vanishes quickly as $k \to \infty$.

Lemma

In each $W_n$ where $1 \leq n \leq m - 1$, both $g_a(z)$ and $E_{\chi_1, \chi_2, k}(z)$ have one zero when $k$ is sufficiently large.
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**Proposition**

When $q_1 > 3$, all these zeros are $\Gamma_0(q_1 q_2)$-inequivalent. That is, there does not exist a $\gamma \in \Gamma_0(q_1 q_2)$ that maps one zero to another.
Theorem

For \( k \) sufficiently large, \( E_{\chi_1, \chi_2, k}(z) \) has \( m = \frac{k}{3} + O\left(\sqrt{k \log^2(k)}\right) \) zeros tending to the vertical line \( \Re(z) = \frac{a + 1/2}{q_2} \) which are \( \Gamma_0(q_1 q_2) \)-inequivalent and become distributed with respect to their angle from the point \( \frac{a}{q_2} \).
The Structure of Zeros

The zeros we have found vary by height on the order of $O\left(\frac{1}{k}\right)$, and there are $\varphi(q_2)$ lines of them in $\left(-\frac{1}{2}, \frac{1}{2}\right)$. 

Conjecture

As $q_2$ tends to infinity, and $k$ tends to infinity much slower, the zeros of $E^{\chi_1,\chi_2,k}(z)$ equidistribute when they are mapped back to the fundamental domain $F$. 

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This means we can try to see them on horocycles, which are horizontal unit length segments in hyperbolic space. We know that horocycles equidistribute.
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Thank you!