On the identification of $k$-inductively pierced codes using toric ideals

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Abstract

Neural codes are binary codes in $\{0, 1\}^n$; here we focus on the ones which represent the firing patterns of a type of neurons called place cells. There is much interest in determining which neural codes can be realized by a collection of convex sets. However, drawing these convex sets, particularly as the number of neurons in a code increases, can be very difficult. It has been shown that an algorithm for drawing Euler diagrams can be used to draw a class of codes that are said to be $k$-inductively pierced for $k = 0, 1, 2$. We use the toric ideal, to show sufficient conditions for a code to be 1- or 2-inductively pierced.

1 Introduction

John O'Keefe’s discovery of place cells earned him the 2014 Nobel Prize for Physiology or Medicine [12]. Place cells are a type of neuron found in certain mammals (including rats, cats, and bats) that help them to locate themselves spatially by firing only when the mammal is in a certain part of its environment. The receptive fields, or areas where the neurons fire, are approximately convex, and can be represented by neural codes. Much study has been done on convex neural codes, specifically focusing on which codes are convex realizable, that is, for which codes one can draw convex receptive fields that correspond to the code [2, 3, 5, 6, 9, 10, 13]. However, if a convex realization exists it can be very difficult to actually draw one. Far less work has been done on how to draw realizations of neural codes. However, [8] have shown that the algorithm created by [14] to draw Euler diagrams can be used to create realizations of codes that are 0-, 1-, or 2-inductively pierced. In this paper, we explore sufficient conditions for 1- and 2-inductively pierced codes. In Section 2, we discuss the definitions necessary for our main results, which we present in Section 3. We conclude with a discussion of our findings in Section 4.

2 Background

We follow the definitions from [8].

Definition 2.1. A neural code on $n$ neurons is a set of binary strings $C \subseteq \{0, 1\}^n$. An element $\sigma$ of $C$ is a codeword.

We will assume that the empty codeword is always in a neural code.
Definition 2.2. A realization of a code $\mathcal{C}$ is a collection of sets $\mathcal{U} = \{U_1, \ldots, U_n\}$ where $U_i \in \mathbb{R}^d$ such that $\mathcal{C} = \mathcal{C}(\mathcal{U}) := \{\sigma \in [n] \mid U_{\sigma} \setminus \bigcup_{j \in [n] \setminus \sigma} U_j \neq \emptyset\}$. A $U_i \in \mathcal{U}$ is a place field of the neuron $i$. A zone in $\mathcal{C}$ is the intersection of a collection of place fields $\{U_i\}$ and each zone corresponds to a codeword.

Definition 2.3. A code $\mathcal{C}$ is convex open if it is realizable by $\mathcal{U} = \{U_1, \ldots, U_n\}$ where all the $U_i$ are convex open sets.

In this paper, we will work under with neural codes that are convex open and realizable in dimension 2. Furthermore, we will assume that our codes are well-formed.

Definition 2.4. A code $\mathcal{C}$ is well-formed if there exists realization of $\mathcal{C}$ such that

- The boundary curves of place fields intersect at only a finite number of points.
- At any given point, at most two boundaries of place fields intersect.
- Each zone is connected.

Example 2.5. As an example, consider the code $\mathcal{C} = \{000, 100, 001, 101, 011, 111\}$. This neural code is convex open and realizable in dimension 2. A well-formed realization of $\mathcal{C}$ is shown in Figure 2. The realization is comprised of 6 zones, one for each codeword, including the empty zone.

Definition 2.6. A $k$-piercing is a place field $U_\ell$ whose boundary intersects the boundaries of $k$ other place fields $U_1, \ldots, U_k$ and adds exactly $2^k$ zones to an existing diagram. A $k$-piercing is identified by a zone in the diagram in which the place field $U_\ell$ is contained.

For example, in Figure 2, $U_2$ is a 1-piercing of $U_1$ identified by the zone 001. Notice that $U_1$ is not a 2-piercing; its boundary intersects the boundaries of two place fields, adding it does not add 4 new regions to the diagram.
Definition 2.7. Let \( C = \{\sigma_1, \ldots, \sigma_m\} \) be a code on \( n \) neurons. \( C \) is \( k \)-inductively pierced if there exists a \( k \)-piercing \( U_\lambda \) in \( C \) such that \( C \setminus \lambda := \{\hat{\sigma}_1, \ldots, \hat{\sigma}_m\} \) is \( k \)-inductively pierced where \( \hat{\sigma}_i = \sigma_i \) except \( \hat{\sigma}_i \) has a 0 in the \( \lambda \)th position.

In this paper will will assume that if a code is \( k \)-inductively pierced, the code is not \((k + 1)\)-inductively pierced.

Definition 2.8. Define \( \phi_c : \mathbb{F}_2[\mathbb{F}_2[p_c | c \in C]] \to \mathbb{F}_2[x_i | i \in [n]] \)

\[ p_c \mapsto \prod_{i \in \text{supp}(c)} x_i. \]

Then, \( I_C := \ker \phi_C \) is the toric ideal of \( C \).

The toric ideal has the computational benefit of being relatively quick to compute using Macaulay2 [7] with the 4ti2 package [1], or with SAGE [4].

In [8], the authors investigated algebraic signatures for 0-, 1-, and 2-inductively pierced codes by considering generators of \( I_C \). In particular, they give necessary and sufficient conditions for 0-inductively pierced codes, along with necessary conditions for 1-inductively pierced codes.

Theorem 2.9 ([8]). Let \( C \) be a well-formed neural code on \( n \) neurons.

1. The neural code \( C \) is 0-inductively pierced if and only if \( I_C = \langle 0 \rangle \).

2. If the neural code \( C \) is 0- or 1- inductively pierced then \( I_C = \langle 0 \rangle \) or generated by quadratics.

3. If the neural code \( C \) contains a triple intersection where the intersections are general, \( I_C \) contains a binomial of degree 3 of particular form, in particular \( p_{111}p_{000} - p_{100}p_{010}p_{001}w \) or \( p_{111}w - p_{100} \ldots w \) where \( v, w \in \{0, 1\}^{n-3} \) correspond to zones in \( C(U) \).

We note that the third claim in Theorem 2.9 includes the case of codes with 2-piercings (Proposition 4.3.1 in [11]) and the case of 2-inductively pierced codes, as summarized by Lemma 2.10.

Lemma 2.10. If a neural code \( C \) contains a 2-piercing or is 2-inductively pierced, then \( I_C \) contains a cubic of form \( p_{111}w^2p_{000} - p_{100}p_{010}p_{001}w \) or \( p_{111}w - p_{100} \ldots w \) where \( v = w \).

We refer to the binomial \( p_{111}w^2p_{000} - p_{100}p_{010}p_{001}w \) for \( w, v \in \{0, 1\}^{n-3} \) as a cubic of a particular form or particular cubic.

In the next section, we investigate sufficient conditions for 1- and 2-inductively pierced codes, and improve upon the second and third statements from Theorem 2.9.

3 Main Results

Throughout the rest of this paper, we assume that \( C = C(U) \) is a convex open neural code on \( n \) neurons that is realizable in dimension 2, so \( U = \{U_1, \ldots, U_n\} \) and the \( U_i \subset \mathbb{R}^2 \).
Proposition 3.1. Let $C$ be a well-formed code on $n$ neurons, and let $I_C$ be its toric ideal. If there exists a cubic generator of $I_C$ of the form $p_{111w}^2 - p_{100w}p_{010w}p_{001v}$ (where $w,v \in \{0,1\}^{n-3}$), then $C\{4,\ldots,n\}$ is 2-inductively pierced.

Proof. The existence of such a cubic in $I_C$ implies that the code contains a triple intersection of $U_1, U_2,$ and $U_3$ along with the associated singletons. Since the code is well-formed, it must also contain the pairwise intersections. Thus, if all other neurons are removed, the remaining code is clearly 2-pierced. \hfill \Box

Corollary 3.2. If $p_{111w}^2 - p_{100w}p_{010w}p_{001v} \in I_C$, then $C$ is not 1-inductively pierced.

Proof. Assume $C$ is 1-inductively pierced. By Proposition 3.1, $C\{4,\ldots,n\}$ contains a 2-piercing. Thus, $U_1$ cannot be a 0- or 1-piercing unless either $U_2$ or $U_3$ is removed. Since the same is true of $U_2$ and $U_3$, none of them can possibly be 0- or 1-piercings. Thus $C$ is not 1-inductively pierced, since as place fields that are 0- or 1-piercings are removed, eventually $C\{4,\ldots,n\}$ must be all that is left. However, $C\{4,\ldots,n\}$ is not 1-inductively pierced, resulting in a contradiction. \hfill \Box

Corollary 3.3. If $p_{111w}^2 - p_{100w}p_{010w}p_{001v} \in I_C$ and $w \neq v$, then $C$ does not contain a 2-piercing, or the 2-piercing is contained within another receptive field and there must exist another cubic with $v = w$.

Proof. If $w \neq v$, then there exists some zone in the potential 2-piercing that is contained in a different zone than other zones in the 2-piercing. Assume that the other zone does not contain an entire receptive field. Then, since the realization is well-formed, $C$ must not contain a 2-piercing. If the other zone does contain an entire receptive field, then the 2-piercing must take place inside this other zone, thus there must also exist a cubic such that $v = w$. \hfill \Box

Lemma 3.4. Let $C$ be a code on $n$ neurons such that in $C \setminus \{3,\ldots,n\}$, $U_1$ is a 1-piercing of $U_2$ and there do not exist $i, j, k \in [n]$ such that in $C \setminus ([n] \setminus \{i,j,k\})$, $U_i$ is a 2-piercing of $U_j$ and $U_k$. Then $C$ is 1-inductively pierced.

Proof. Suppose towards contradiction that $C$ is not 1-inductively pierced. Then, there must be some neuron $s$ such that $U_s$ intersects both $U_1$ and $U_2$ such that fewer than 4 zones are created and neither $U_1$ nor $U_2$ is a 2-piercing of $U_s$. However, this can only happen if $C$ is not well-formed. However, we assumed that $C$ is well-formed, resulting in a contradiction. \hfill \Box

Theorem 3.5. Let $C$ be a well-formed code on $n$ neurons, and let $I_C$ be its toric ideal. If there exists a cubic of the form $p_{111w}^2 - p_{100w}p_{010w}p_{001v} \in I_C$ and in all such cases $w = v$, then $C$ is 2-inductively pierced.

Proof. We will proceed by proving the contrapositive. That is, we will assume that $C$ is not 2-inductively pierced and show that whenever the toric ideal contains cubics of the particular form, there exists one such cubic with $w \neq v$.

By Proposition 3.1 $C\{4,\ldots,n\}$ is 2-inductively pierced, so without loss of generality, $U_1, U_2, U_3$ are all two-piercings of each other.

Suppose that there exists some $U_j$ that obstructs $U_1, U_2,$ and $U_3$ from being 2-piercings of each other in $C$. If $U_j$ contains the triple intersection, then all zones other than the triple intersection are partially contained in $U_j$. 


If $U_j$ does not contain the triple intersection, then $U_j$ partially contains it. In either case, we have the zones necessary to construct a cubic with $v \neq w$, as illustrated in Figure 3.

Furthermore, using the proof of Lemma 3.4, the only way for $C$ to fail to be 2-inductively pierced is if there exists such a $U_j$. That is to say, there must be a curve $U_j$ that obstructs $U_1$, $U_2$, and $U_3$ from all being two piercings of each other, so $U_j$ cannot be a 0- or 1-piercing.

Thus, there must exist a cubic of form $p_{111}w^2p_000v - p_{100}w^2p_100p_{001}v \in I_C$ where $v \neq w$.

**Proposition 3.6.** Let $C$ be a well formed code on $n$ neurons. If there is no cubic of the form $p_{111}w^2p_000v - p_{100}w^2p_100p_{001}v \in I_C$ and $I_C \neq \langle 0 \rangle$, then $C$ is 1-inductively pierced.

**Proof.** The lack of a cubic of form $p_{111}w^2p_000v - p_{100}w^2p_100p_{001}v \in I_C$ implies that there do not exist $i, j, k \in [n]$ such that in $C \setminus ([n] \setminus \{i, j, k\})$, $U_i$ is a 2-piercing of $U_j$ and $U_k$. Since $I_C \neq \langle 0 \rangle$, $C$ cannot be 0-inductively pierced, thus there must be some curves that intersect. Thus by Lemma 3.4 $C$ must be 1-inductively pierced. \qed

![Figure 3](image-url)

Figure 3: Examples of codes that are not 2-inductively pierced and have cubics of particular form where $v \neq w$ in $I_C$. Note that on the left, $U_4$ contains the entire triple intersection $U_1 \cap U_2 \cap U_3$, and on the right $U_4$ partially contains the triple intersection.

The existence of particular cubic binomials in the toric ideal can tell us much about the code. However, computing the entire toric ideal is computationally expensive; we would much rather only look at generating sets. Unfortunately, the existence of particular cubics in the toric ideal does not imply they are generators.

**Example 3.7.** As discussed in [11], one generating set for the toric ideal of the code $A_1 = \{000, 100, 010, 001, 110, 101, 011, 111\}$ is $\{p_{111} - p_{100}p_{010}p_{001}; p_{110} - p_{100}p_{010}; p_{101} - p_{100}p_{001}; p_{011} - p_{010}p_{001}\}$. This generating set has a particular cubic.
Another generating set for \( I_{A_1} \), which consists only of quadratics is \( \{p_{110} - p_{100}p_{010}, \ p_{101} - p_{100}p_{001}, \ p_{011} - p_{010}p_{001}, \ p_{111} - p_{110}p_{001}\} \).

From the second generating set for \( I_{A_1} \) in Example 3.7, we notice that a priori we may not immediately see the existence of a particular cubic from the generators directly. However, there is a class of quadratics that can imply the existence of the particular cubics.

**Definition 3.8.** Let \( \mathcal{C} \) be code on \( n \) neurons and \( w, v \in \{0,1\}^{n-3} \). A pair of quadratics of a particular form

\[
\begin{align*}
\{p_{111}w p_{000}v - p_{110}w p_{001}v, \quad &p_{110}w p_{000}v - p_{100}w p_{010}v \} \\
&\text{or} \quad \{p_{111}w p_{000}v - p_{101}w p_{010}v, \quad &p_{101}w p_{000}v - p_{100}w p_{001}v \}
\end{align*}
\]

are called **friendly quadratics**.

**Proposition 3.9.** Let \( z \in \{0,1\}^{n-3} \). If any of the friendly quadratics are in the generating set for the toric ideal \( I_{\mathcal{C}} \) of \( \mathcal{C} \), then \( I_{\mathcal{C}} \) contains a cubic of the form

\[
p_{111}z^2 - p_{100}z p_{010}z \in I_{\mathcal{C}}.
\]

*Proof.* This is straightforward to see since \( I_{\mathcal{C}} \) is an ideal, hence is closed under multiplication by elements of the ring. Consider the first case. Since \( p_{111}w p_{000}v - p_{110}z p_{001}z \in I_{\mathcal{C}} \), we get that \( p_{000}z, p_{001}z \in \mathbb{F}_2[p_c \mid c \in \mathcal{C}] \) and so,

\[
p_{111}z^2 - p_{100}z p_{010}z p_{001}z = p_{000}z(p_{111}z p_{000}z - p_{110}z p_{001}z) + p_{001}z(p_{110}z p_{000}z - p_{100}z p_{010}z) \in I_{\mathcal{C}}.
\]

□

**Example 3.10.** Consider the following neural code on 5 neurons, \( \mathcal{C} = \{00001, 10001, 01001, 00011, 11001, 10011, 01011, 00111, 11011, 10111, 01111, 11111\} \). One generating set for \( I_{\mathcal{C}} \) is

\[
\langle p_{00111}p_{11111} - p_{10111}p_{01111}, \ p_{01011}p_{11111} - p_{11011}p_{01111}, \ p_{10011}p_{11111} - p_{11011}p_{10111}, \ p_{00011}p_{11011} - p_{10011}p_{00111}, \ p_{11011}p_{10111} - p_{11011}p_{01111}, \ p_{01101}p_{10111} - p_{11101}p_{10011}, \ p_{10101}p_{11011} - p_{11101}p_{10011}, \ p_{01001}p_{11011} - p_{11001}p_{10111}, \ p_{10001}p_{11111} - p_{11001}p_{10111}, \ p_{00101}p_{11111} - p_{10101}p_{01111}, \ p_{11001}p_{10111} - p_{11001}p_{10011}, \ p_{11101}p_{10011} - p_{10101}p_{01111}, p_{00001}p_{11111} - p_{10001}p_{10011}, \ p_{11001}p_{10011} - p_{11001}p_{00111}, \ p_{00001}p_{10011} - p_{10001}p_{00111}, \ p_{00001}p_{11011} - p_{10001}p_{11011}, \ p_{00001}p_{11111} - p_{10001}p_{11111} \rangle.
\]

Note that no cubics appear in this generating set. However, \( p_{00001}p_{11111} - p_{11001}p_{00111} \) and \( p_{00001}p_{10011} - p_{10001}p_{10001} \) do. These are a pair of friendly quadratics, thus a cubic of particular form must appear in \( I_{\mathcal{C}} \), namely a particular cubic with \( w = v \). We see from a realization of \( \mathcal{C} \) in Figure 4 that this code is 2-inductively pierced, even though it does not satisfy the conditions of Theorem 3.5.
4 Discussion

We have shown sufficient conditions for both 1- and 2-inductively pierced codes. However, these conditions possess significant weaknesses. Our sufficient condition for a code to be 1-inductively pierced relies on being able to analyze the entire toric ideal, a task that becomes unfeasible as for codes on $N$ neurons, for large $N$. However, analyzing generating sets of the toric ideal is much easier. If we could classify all possible ways that a cubic of particular form can be generated, we could identify whether a cubic of particular form is in the toric ideal simply by looking an arbitrary generating set. If this were accomplished, Theorem 3.5 and Proposition 3.6 would become much more powerful.

Another direction for future study lies in identify which receptive fields could potentially form a 2-piercing. In this paper we assumed that we knew which three place fields were involved in a 2-piercing just from looking at the code. One could simply check every possible combination of three neurons, but as the number of neurons increases, this becomes far more difficult.

Further research in identifying which codes are well-formed and realizable in two dimensions would also be of great benefit. We have made the assumption that the codes we work with are well-formed and realizable in two dimensions, but as of yet we are unaware of the existence of any tools to determine these conditions.
Figure 5: A wombat, which may or may not have place cells, though linguistic analysis suggests the affirmative.

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References

[1] 4ti2 team. 4ti2—a software package for algebraic, geometric and combinatorial problems on linear spaces. Available at www.4ti2.de.


