Inductively Pierced Codes and Toric Ideals

Samuel Muthiah

Westmont College

July 17, 2017
Place Cells

- In 2014 John O’Keefe received the Nobel Prize for his discovery of place cells.
- Place cells are part of the way certain mammals’ brains identify where there are spatially.
- Place cells fire in approximately convex regions.

Figure: Place Cells
Neural Codes

Definition

A neural code on $n$ neurons is a set of binary strings $\mathcal{C} \subseteq \{0, 1\}^n$. The elements of $\mathcal{C}$ are called codewords.

$$\mathcal{C} = \{000, 100, 001, 101, 011, 111\}$$
Neural Codes

Definition

A realization of a code $\mathcal{C}$ on $n$ neurons is a collection of sets $\mathcal{U} = \{U_1, \ldots, U_n\}$ such that $\mathcal{C}(\mathcal{U}) = \mathcal{C}$.

$$\mathcal{C} = \{000, 100, 001, 101, 011, 111\}$$
Neural Codes

**Definition**

A code $\mathcal{C}$ is *convex* if there exists a realization of $\mathcal{C}$ by convex sets.

$$\mathcal{C} = \{000, 100, 001, 101, 011, 111\}$$

![Diagram of a code realization with symbols 1, 2, and 3]
Neural Codes

Definition

The realization of a code $C$ is **well-formed** if

- Curves intersect at a finite number of points
- At any given point, at most two curves intersect
- Each zone is connected

$C = \{000, 100, 001, 101, 011, 111\}$
Need for Algorithms

\{000000, 100000, 010000, 001000, 000100, 000010, 110000, 100010, 011000, 010100, 010010, 001100, 000110, 000011, 110010, 011100, 010110\}
Need for Algorithms

\{000000, 100000, 010000, 001000, 000100, 000010, 110000, 100010, 011000, 010100, 010010, 001100, 000110, 000011, 110010, 011100, 010110}\}
**k-Piercings**

**Definition**

A **k-piercing** is a curve that pierces (intersects) \( k \) other curves and that adds \( 2^k \) zones when added to an existing diagram.
**k-Piercings**

**Definition**

A **k-piercing** is a curve that pierces (intersects) \( k \) other curves and that adds \( 2^k \) zones when added to an existing diagram.

**Figure:** Example of a 1-Piercing

**Figure:** Example of a 2-Piercing
k-Inductively Pierced

Definition

A neural code $C$ is **k-inductively pierced** if $C$ has a $0$-, $1$-, ..., or $k$-piercing $\lambda$ and $C - \lambda$ is $k$-inductively pierced.
k-Inductively Pierced

Definition

A neural code $C$ is **k-inductively pierced** if $C$ has a $0-$, $1-$, ..., or $k-$ piercing $\lambda$ and $C - \lambda$ is $k$-inductively pierced.
Toric Ideals

- Let \( C = \{c_1, \ldots, c_m\} \) be a neural code on \( n \) neurons
- Let \( \phi_c : \mathbb{K}[p_c|c \in C\setminus(0, \ldots, 0)] \rightarrow \mathbb{K}[x_i|i \in [n]] \)

\[
p_c \mapsto \prod_{i \in \text{supp}(c)} x_i
\]
Toric Ideals

- Let $\mathcal{C} = \{c_1, \ldots, c_m\}$ be a neural code on $n$ neurons
- Let $\phi_c : \mathbb{K}[p_c | c \in \mathcal{C}\backslash(0, \ldots, 0)] \to \mathbb{K}[x_i | i \in [n]]$

\[ p_c \mapsto \prod_{i \in \text{supp}(c)} x_i \]

**Definition**

The toric ideal of the neural code $\mathcal{C}$ is $I_{\mathcal{C}} := \ker \phi_c$

\[ p_{101}p_{110} - p_{111}p_{100} \mapsto x_1x_3 \cdot x_1x_2 - x_1x_2x_3 \cdot x_1 = 0 \]
Identifying $k$-Piercings

**Theorem (Gross-Obatake-Youngs)**

Let $\mathcal{C}$ be well formed.

- **The neural code** $\mathcal{C}$ **is 0-inductively pierced if and only if** $I_\mathcal{C} = \langle 0 \rangle$.
- **If the neural code** $\mathcal{C}$ **is 0- or 1- inductively pierced then** $I_\mathcal{C} = \langle 0 \rangle$ **or** generated by quadratics.
- **If the neural code** $\mathcal{C}$ **has a 2-piercing then** $I_\mathcal{C}$ **contains a binomial of degree 3 of particular form, in particular** $p_{111w}p_{000v}^2 - p_{100v}p_{010v}p_{001w}$ **or** $p_{111w} - p_{100...0}p_{010...0}p_{001w}$ **where** $v, w$ **are zones in** $\mathcal{C}(U)$.
Identifying k-Piercings

Take the code $C = \{0001, 1001, 0101, 0011, 1101, 1011, 0111, 1111\}$.

One set of generators of its toric ideal is:

$$\langle -p_{1011}p_{0111} + p_{0011}p_{1111}, -p_{1101}p_{0111} + p_{0101}p_{1111},$$
$$-p_{0101}p_{1011} + p_{1001}p_{0111}, -p_{0011}p_{1101} + p_{1001}p_{0111},$$
$$-p_{1011}p_{1011} + p_{1001}p_{1111}, -p_{1001}p_{0101} + p_{0001}p_{1101},$$
$$-p_{1001}p_{0011} + p_{0001}p_{1011}, -p_{1001}p_{0111} + p_{0001}p_{1111},$$
$$-p_{0101}p_{0011} + p_{0001}p_{0111} \rangle.$$
Identifying k-Piercings

Take the code $\mathcal{C} = \{0001, 1001, 0101, 0011, 1101, 1011, 0111, 1111\}$. 

Figure: Realization of $\mathcal{C}$
Identifying k-Piercings

\[ \langle -p_{1011}p_{0111} + p_{0011}p_{1111}, -p_{1101}p_{0111} + p_{0101}p_{1111}, -p_{0101}p_{1011} + p_{1001}p_{0111}, -p_{0011}p_{1101} + p_{1001}p_{0111}, -p_{1011}p_{1011} + p_{1001}p_{1111}, -p_{1001}p_{0101} + p_{0001}p_{1101}, -p_{1001}p_{0011} + p_{0001}p_{1011}, -p_{1001}p_{0111} + p_{0001}p_{1111}, -p_{0101}p_{0011} + p_{0001}p_{0111} \rangle \]
Constructing Cubics

\[ p_{0001}p_{1111} - p_{1001}p_{0111}, \quad p_{0001}p_{0111} - p_{0101}p_{0011} \in I_C \]

\[ p_{0001}(p_{1111}p_{0001} - p_{1001}p_{0111}) + p_{1001}(p_{0111}p_{0001} - p_{0101}p_{0011}) \in I_C \]
Constructing Cubics

\[ p_{0001}p_{1111} = p_{1001}p_{0111}, \]
\[ p_{0001}p_{0111} = p_{0101}p_{0011} \in I_C \]
\[ \downarrow \]
\[ p_{0001}(p_{1111}p_{0001} - p_{1001}p_{0111}) + p_{1001}(p_{0111}p_{0001} - p_{0101}p_{0011}) \in I_C \]

and

\[ p_{0001}(p_{1111}p_{0001} - p_{1001}p_{0111}) + p_{1001}(p_{0111}p_{0001} - p_{0101}p_{0011}) = p_{1111}p_{0001}^2 - p_{1001}p_{0101}p_{0011} \]
Special Quadratics

\[ p_{000v}(p_{111w}p_{000v} - p_{110v}p_{001w}) + p_{001w}(p_{110w}p_{000v} - p_{100v}p_{010v}) \]
\[ p_{000v}(p_{111w}p_{000v} - p_{101v}p_{010w}) + p_{010w}(p_{101w}p_{000v} - p_{100v}p_{001v}) \]
\[ p_{000v}(p_{111w}p_{000v} - p_{011v}p_{100w}) + p_{001w}(p_{011w}p_{000v} - p_{010v}p_{001v}) \]
Special Quadratics

\[ p_{000v}(p_{111w}p_{000v} - p_{110v}p_{001w}) + p_{001w}(p_{110w}p_{000v} - p_{100v}p_{010v}) \]
\[ p_{000v}(p_{111w}p_{000v} - p_{101v}p_{010w}) + p_{010w}(p_{101w}p_{000v} - p_{100v}p_{001v}) \]
\[ p_{000v}(p_{111w}p_{000v} - p_{011v}p_{100w}) + p_{001w}(p_{011w}p_{000v} - p_{010v}p_{001v}) \]
Sufficient Condition?

Take the code

\[ C = \{1000, 0100, 0010, 1100, 1010, 1001, 0110, 0101, 0011, 1101, 1011, 0111, 1111\} \]

\[ p_{1111} - p_{1000} p_{0100} p_{0011} \in I_C \]
Theorem (Hoch-M.-Obatake)

Let $C$ be a well-formed code, and let $I_C$ be its toric ideal. If there exists a cubic generator of $I_C$ of the form $p_{111}w^2p_{000}^2 - p_{100}wp_{010}v^2p_{001}v$, then $C(\mathcal{U})\backslash\bigcup_{i=4}^m U_i$ is 2-inductively pierced.

Corollary

If $p_{111}wp_{000}^2 - p_{100}wp_{010}v^2p_{001}v \in I_C$, then $C$ is not 1-inductively pierced.
Example

\[ p_{111w}p_{000v}^2 - p_{100v}p_{010v}p_{001w} \]
\[ p_{111w} - p_{100...0}p_{010...0}p_{001w} \]
Example

\[
p_{111w}p_{000w}^2 - p_{100w}p_{010w}p_{001w}
\]

\[
p_{111w} - p_{100...0}p_{010...0}p_{001w}
\]

3

1

2
Example

\[ p_{111w} p_{000v}^2 - p_{100v} p_{010v} p_{001w} \]
\[ p_{111w} - p_{100...0} p_{010...0} p_{001w} \]
Example

\[ p_{111}^2 p_{000}^2 - p_{100}^2 p_{010}^2 p_{001}^2 \]

\[ p_{111} - p_{100}^2 p_{010}^2 p_{001}^2 \]
Can we classify all possible ways of generating the cubics of a particular form so we can identify a 1-piercing from any generating set of the toric ideal?
Can we classify all possible ways of generating the cubics of a particular form so we can identify a 1-piercing from any generating set of the toric ideal?

Which codes realizable in 2 dimensions are well-formed?
Discussion

- Can we classify all possible ways of generating the cubics of a particular form so we can identify a 1-piercing from any generating set of the toric ideal?
- Which codes realizable in 2 dimensions are well-formed?
- Can we identify which neurons potentially form a 2-piercing?
Acknowledgments

Special thanks to
- Molly Hoch
- Dr. Anne Shiu
- Nida Obatake
- Ola Sobieska
- Jonathan Tyler
- The National Science Foundation
- Texas A&M University
Thank You!