On Classification of the Unitarizability of Irreducible Representations of $B_5$

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I got 99 problems...

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I wanted to classify representations of $B_5$ with dimension greater than 5. This means being able to write down the form of the matrices for this representation.
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The Problem
I wanted to classify representations of $B_5$ with dimension greater than 5. This means being able to write down the form of the matrices for this representation.

The Strategy
I needed to find a special basis in which all the matrices of this representation acquire a predetermined form.
**PLOT TWIST!**

All of my approaches to the problem from the previous slide failed!
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1. With two weeks left, Small Paul and I joined forces!
2. We successfully classified which representations of $B_5$ of dimension $d \leq 5$ are unitarizable!
Unitarizability

1. In order to build a functioning quantum computer, we need to be able to manipulate quantum information, the fundamental unit of which is the qubit.
2. A qubit may be represented as a vector in a complex Hilbert space.
3. We can manipulate this quantum information by applying a unitary transformation (matrix).

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What Words Mean

**Definition (Braid Group)**

The **braid group on** $n$-*strands* is given by

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \forall \ i \in \{1, \ldots n - 1\} \rangle$$

$$\sigma_i \sigma_j = \sigma_i \sigma_{j} \quad \forall \ |i - j| \neq 1$$
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Definition (Representation)

A **representation** of a group \( G \) is a pair \((\rho, V)\), where \( V \) is a \( d \) dimensional vector space over \( \mathbb{C} \) and \( \rho \) is a group homomorphism from \( G \) to the collection of \( d \times d \) invertible matrices over \( \mathbb{C} \).
Definition (Irreducible)

A representation is irreducible if \( V \) contains no proper, non-trivial subspaces \( W \) such that \( \rho(g)w \in W \) for all \( g \in G, w \in W \).
What Even More Words Mean

Definition (Irreducible)
A representation is **irreducible** if $V$ contains no proper, non-trivial subspaces $W$ such that $\rho(g)w \in W$ for all $g \in G$, $w \in W$.

Definition (Unitarizable)
A representation $\rho$ is **unitarizable** provided there exists a Hermitian inner product $\langle \cdot | \cdot \rangle_A$ such that $\langle \rho(g)v | \rho(g)w \rangle_A = \langle v | w \rangle_A$ for all $g \in G$ and for all $v, w \in V$. 

Note: The arbitrary inner product $\langle \cdot | \cdot \rangle_A$ may be related to the standard inner product via $\langle v | w \rangle_A = \langle Av | w \rangle_A$ for some matrix $A$. 
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Quick Example

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Under the standard inner product, a unitary matrix $\rho(g)$ has the property that $\langle \rho(g)v|\rho(g)v \rangle = \langle v|v \rangle$. 
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Under the standard inner product, a unitary matrix $\rho(g)$ has the property that $\langle \rho(g) v | \rho(g) v \rangle = \langle v | v \rangle$.

In other words, applying a unitary matrix to a vector does not change the vector’s length!
Definition (Adjoint)

Let $A$ be a matrix, then we define the adjoint of $\rho(g)$ with respect to $A$ via $\rho(g)^* = A^{-1} \rho(g)^\dagger A$, where $^\dagger$ denotes complex conjugate transpose.

Useful Tools

**Definition (Adjoint)**

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**Definition (Unitarizable Matrix)**

A matrix $\rho(g)$ is unitarizable provided there exists a matrix $A$ such that $\rho(g)\rho(g)^* = \rho(g)A^{-1}\rho(g)^\dagger A = I$. 
The Classification Problem

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Classification

To classify the unitarizability of the representations of $B_5$, we need to check the unitarizability of $\tilde{\rho} = \chi(c) \otimes \rho(t)$ given $\rho(t)$. 
The Process

Consider the representation \( \tilde{\rho} = \chi(c) \otimes \rho(t) \) of \( B_5 \). It follows from the definition that \( \tilde{\rho} \) is unitarizable if and only if there exists an \( A \) such that,
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After further manipulation, we see that the above is equivalent to

$$0 = A\tilde{\rho}(\sigma_i) - ((\tilde{\rho}(\sigma_i))^\dagger)^{-1}A$$

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= A(c\rho(t)(\sigma_i)) - ((c\rho(t)(\sigma_i))^{\dagger})^{-1} A \\
= c(A\rho(t)(\sigma_i)) - \frac{1}{\bar{c}}((\rho(t)(\sigma_i))^{\dagger})^{-1} A \\
= c\bar{c}(A\rho(t)(\sigma_i)) - ((\rho(t)(\sigma_i))^{\dagger})^{-1} A
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We see then that if \( c\bar{c} = 1 \), i.e. if \( c \) is on the unit circle, then \( \dot{\rho} \) is unitarizable exactly when \( \rho(t) \) is.
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We see then that if $c\bar{c} = 1$, i.e. if $c$ is on the unit circle, then $\dot{\rho}$ is unitarizable exactly when $\rho(t)$ is.

An interesting question is whether there exists some $c$ and some non-unitarizable representation $\rho(t)$ such that $\tilde{\rho}$ is unitarizable.
Results

1. Given $\rho(t)$, I set up some MatLab code which converts the equation matrix

$$0 = c\bar{c}(A\rho(t)(\sigma_i)) - ((\rho(t)(\sigma_i))^\dagger)^{-1}A$$

into a master coefficient matrix composed of the coefficient matrices for each $\sigma_i$. 

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2. I then solved the coefficient matrices for the Hecke $\rho(t) = H(t)$, and reduced-extended Burau $\rho(t) = \hat{\beta}(t)$ representations.
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2. I then solved the coefficient matrices for the Hecke $\rho(t) = H(t)$, and reduced-extended Burau $\rho(t) = \hat{\beta}(t)$ representations.

3. I found that for both $H$ and $\hat{\beta}$ there was no $c$ that satisfied the above equation for all $\sigma_i$.

4. Collectively, Small Paul and I have fully classified which representations of $B_5$ of dimension $d \leq 5$ are unitarizable!
Next Steps

1. Now that we are done with the representations of $B_5$, Paul and I have ambitions to classify representations of $B_n$ for $n \neq 5$. 

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1. Now that we are done with the representations of $B_5$, Paul and I have ambitions to classify representations of $B_n$ for $n \neq 5$.

2. In this process, if we do not find any non-unitarizable representations $\rho(t)$ that can be unitarized with the right $\chi(c)$ then we will have shown by exhaustion that $\tilde{\rho}$ is unitarizable if and only if $c$ is on the unit circle and $\rho(t)$ is unitarizable.
Thanks for listening!

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