A Faster Randomized Algorithm for Counting Roots in \( \mathbb{Z}/(p^k) \)

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Today we will...

- Review the randomized algorithm for counting mod $p^k$
- See some specific examples regarding computational time
- Go over the complexity bound of the algorithm
- Discuss a bound on the number of roots mod $p^k$
Counting roots in $\mathbb{Z}/(p)[x]$ is easy if you allow randomization, thanks to the work of Zieler, Berlekamp, and many others, dating back to the 1960’s.

These methods take advantage of $\mathbb{Z}/(p)[x]$ being a unique factorization domain.
Factorization

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  - One simple method: compute the gcd($x^p - x, f$) in $\mathbb{Z}/(p)$
  - By Fermat’s Little Theorem, we know that for a prime $p$, $x^p - x = x(x - 1) \cdots (x - (p - 1)) \mod p$
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Consider the Taylor expansion of a polynomial $f \in \mathbb{Z}[x]$ of degree $d$, where $\zeta \in \mathbb{Z}$ is a root of the mod $p$ reduction of $f$ and $\varepsilon \in \{0, \ldots, p^k - 1\}$:

$$f(\zeta + p\varepsilon) = f(\zeta) + f'(\zeta)p\varepsilon + \cdots + \frac{1}{(k-1)!} f^{(k-1)}(\zeta)(p\varepsilon)^{k-1} \mod p^k.$$
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Definition: Let $s \in \{1, \ldots, k\}$ be the maximal integer such that $p^s$ divides each of $f(\zeta), \ldots, \frac{1}{(k-1)!}f^{(k-1)}(\zeta)(p\epsilon)^{k-1}$. 

Lemma (Hensel's Lemma) If $f \in \mathbb{Z}[x]$ is a polynomial with integer coefficients, $p$ is prime, and $\zeta \in \{0, \ldots, p^k - 1\}$ is a root of $f \mod p^k$ and $f'(\zeta) \not\equiv 0 \mod p^k$, then there is a unique $\zeta \in \{0, \ldots, p^k + 1\}$ with $f(\zeta) \equiv 0 \mod p^{k+1}$ and $\zeta \equiv \zeta \mod p^k$. 

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**Lemma (Hensel’s Lemma)**

If \( f \in \mathbb{Z}[x] \) is a polynomial with integer coefficients, \( p \) is prime, and \( \zeta_J \in \{0, \ldots, p^{J-1} - 1\} \) is a root of \( f \) (mod \( p^J \)) and \( f'(\zeta_J) \neq 0 \) (mod \( p \)), then there is a unique \( \zeta \in \{0, \ldots, p^{J+1} - 1\} \) with \( f(\zeta) = 0 \) (mod \( p^{J+1} \)) and \( \zeta = \zeta_J \) (mod \( p^J \)).
Overview of Algorithm

For each root $\zeta$ of the mod $p$ reduction of $f$, we have the following:

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When $s \in \{2, \ldots, k - 1\}$, we can reapply the algorithm to an instance of counting roots for the polynomial $f_{\zeta}(\varepsilon) = f(\zeta) p^s + f'(\zeta) p^{s-1} \varepsilon + \cdots + f(k-1)(\zeta) (p-1) \varepsilon^{k-1}$ in $\mathbb{Z}/(p^k-s)$. 

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  $$f_\zeta(\varepsilon) = \frac{f(\zeta)}{p^s} + \frac{f'(\zeta)}{p^{s-1}} \varepsilon + \cdots + \frac{f^{(k-1)}(\zeta)}{(k-1)! p^{s-(k-1)}} \varepsilon^{k-1} \text{ in } \mathbb{Z}/(p^{k-s}).$$

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We use Kedlaya-Umans fast $\mathbb{Z}/(p)[x]$ factoring algorithm, which takes time $d^{1.5+o(1)}(\log p)^{1+o(1)} + d^{1+o(1)}(\log p)^{2+o(1)}$ for a degree $d$ polynomial.
Ideas Behind Proof of Complexity Bound

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- To simplify, the complexity is less than or equal to (the number of nodes in the recursion tree)(the complexity of factoring over $(\mathbb{Z}/(p))[x]$).
- The depth and branching of our recurrence tree is strongly limited by the value of $k$.
- Optimizing parameters, the worst case is when $d \approx e \approx 2.71828$ and the depth is $\frac{k}{e}$. 
The randomized algorithm counts the number of roots in time $d^{1.5+o(1)}(\log p)^{2+o(1)}(1.12)^k$. This complexity bound refers to counting roots, not to finding/storing them; we can have over $\lfloor d^k \rfloor p^{k-1}$ roots, so we cannot achieve this same time bound if we store every single root. Example: The polynomial $f(x) = (x-2)^7(x-1)^3$ with $p=17$, $k=7$ has $24,221,090$ roots (this is greater than $\lfloor d^k \rfloor p^{k-1} = 24,137,569$). Computing this number of roots took .004 seconds, but storing and listing them all would take much longer. (Counting the number of roots using brute force took 39 minutes.)
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Comparing Methods

- The randomized algorithm counts the number of roots in time $d^{1.5+o(1)}(\log p)^2+o(1)(1.12)^k$.
- In comparison, brute force counting takes time $\approx p^k$.
- We expect the randomized algorithm to be faster even for $p$ as small as 2 ($1.12^k$ vs. $2^k$).
The table below shows the average difference in computation time for the number of roots of 100 random polynomials of degree less than or equal to 100 in $\mathbb{Z}/(2^k)$ for the given $k$, between brute force and the randomized algorithm (negative implies brute force was faster):

<table>
<thead>
<tr>
<th>$k$</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg Diff (in seconds)</td>
<td>-0.0011</td>
<td>-0.00029</td>
<td>0.0028</td>
<td>0.01701</td>
<td>0.32499</td>
</tr>
</tbody>
</table>
Timing Data

- We expect the randomized algorithm to take the longest when a polynomial has many degenerate roots because a polynomial of this type will require many recursive calls.
- Polynomials with many degenerate roots do take longer than a random polynomial, but overall the randomized algorithm still outperforms other methods.

Example: Counting roots of $f(x) = (x - 1)(x - 2)^2 \cdots (x - 10)^{10}$ in $\mathbb{Z}/(31^{10})$ took 6.4 seconds using the randomized algorithm. For comparison, a random polynomial of the same degree (55) took 1 millisecond with the randomized algorithm, and counting roots using brute force for just $31^{6}$ took 2.7 hours.
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For comparison, a random polynomial of the same degree (55) took 1 millisecond with the randomized algorithm, and counting roots using brute force for just \( 31^6 \) took 2.7 hours.
The number of roots of a polynomial in $\mathbb{Z}/(p^k)$ can be very large, especially for polynomials with many degenerate roots mod $p$.

Example: $(x - 2)^{50}$ has 5132842958629010337866366828195 (31 digit number) roots in $\mathbb{Z}/(25^{29})$. 
The number of roots of a polynomial in $\mathbb{Z}/(p^k)$ can be very large, especially for polynomials with many degenerate roots mod $p$.

Example: $(x - 2)^{50}$ has $5132842958629010337866366828195$ (31 digit number) roots in $\mathbb{Z}/(25^{29})$.

But we do have an upper bound on the number of roots based on the sizes of $k$, $p$ and $d$. 
Lemma

If a root $\zeta$ of the mod $p$ reduction of $f$ has multiplicity $j$, then $s_\zeta \leq j$, where $s_\zeta$ is the greatest integer such that $p^{s_\zeta}$ divides each of $f(\zeta), \ldots, \frac{f^{(k-1)}(\zeta)}{(k-1)!} p^{k-1} \epsilon_{k-1}$. 

Proof: If $\zeta$ has multiplicity $j$, then $f(\zeta) = \cdots = f^{j-1}(\zeta) = 0 \pmod{p}$, but $f^{(j)}(\zeta) \neq 0 \pmod{p}$. So $\frac{f^{(j)}(\zeta)}{j!} p^j$ is divisible by $p^j$ but not $p^{j+1}$ and therefore $s_\zeta \leq j$. 

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This is important because it tells us that we can have at most $\left\lfloor \frac{d}{s} \right\rfloor$ roots of $f \pmod{p}$ for each $s$. 
Theorem

Let $p$ be a prime, $f \in \mathbb{Z}[x]$ a polynomial of degree $d$, and $k \in \mathbb{N}$ such that $d \geq k \geq 2$. Then the number of roots of $f$ in $\mathbb{Z}/(p^k)$ is less than or equal to $\min\{d, p\} p^{k-1}$. 

We know there are polynomials with more than $\min\{\lfloor d/k \rfloor, p\} p^{k-1}$ roots in $\mathbb{Z}/(p^k)$, so our bound is within a factor of $k$ of optimality.
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Root Bound Examples

Polynomials with $d \geq p$ having $p^k$ roots in $\mathbb{Z}/(p^k)$:

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2. $g(x) = (x^{p^k-p^{k-1}} - 1)x^k$ has degree $p^k - p^{k-1} + k$ and also vanishes on all of $\mathbb{Z}/(p^k)$. 

Root Bound Examples
When $k = 2$, there are a maximum of $\min\{\lfloor \frac{d}{2} \rfloor, p\} p + (d \bmod k)$ roots in $\mathbb{Z}/(p^2)$.

This upper bound is sharp. For example, with $p = 5$ the degree 3 polynomial $(x - 1)^2 x$ has $\lfloor \frac{3}{2} \rfloor \cdot 5 + (3 \bmod 2) = 6$ roots in $\mathbb{Z}/(p^2)$. 
Conclusions

- We have a Las Vegas randomized algorithm for counting the number of roots of a degree $d$ polynomial $f \in \mathbb{Z}[x]$ in $\mathbb{Z}/(p^k)$.
- The complexity of the randomized algorithm is given by $O(1.12^k)$.
- We see time improvements for computations using the randomized algorithm over brute-force counting even for $p = 2$.
- An upper bound on the number of roots is $\min\{d, p\}p^{k-1}$, and a sharp upper bound for $k = 2$ is given by $\min\{\left\lfloor \frac{d}{2} \right\rfloor, p\}p + (d \mod k)$. 