Dedekind Sums Arising from Generalized Eisenstein Series
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Abstract

Given primitive Dirichlet characters $\chi_1$ and $\chi_2$, we study the weight zero Eisenstein series $E_{\chi_1,\chi_2}(z,s)$ at $s = 1$. We examine transformation properties of terms arising from the Fourier expansion of the Eisenstein series, and we express these properties with a generalized Dedekind sum formula in terms of Bernoulli functions.

1 Introduction

Let $\chi_1, \chi_2$ be primitive Dirichlet characters modulo $q_1, q_2$, respectively, with $\chi_1(-1)\chi_2(-1) = -1$. We investigate the generalized Dedekind sum arising from the weight-zero Eisenstein series attached to characters. The Eisenstein series is defined as

$$E_{\chi_1,\chi_2}(z,s) = \frac{1}{2} \sum_{(c,d)=1} \frac{(q_2y)^s\chi_1(c)\chi_2(d)}{|cq_2z + d|^{2s}}.$$  

Here $E_{\chi_1,\chi_2}$ is an automorphic form on the congruence subgroup $\Gamma_0(q_1q_2)$ of nebentypus $\psi = \chi_1\chi_2$. Precisely, for all $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(q_1q_2)$, $E_{\chi_1,\chi_2}(\gamma z, s) = \psi(\gamma) E_{\chi_1,\chi_2}(z, s)$, where $\psi(\gamma) = \psi(d)$.

We are investigating the Eisenstein series at $s = 1$. Historically, many authors have been interested in the Eisenstein series as $s \to 1$ because it has been a way to arrive at the Dedekind sum. Additionally, the first Kronecker Limit Formula involves looking at $\lim_{s \to 1}$ of the Eisenstein series with trivial Dirichlet characters, and investigating the Eisenstein series at $s = 1$ leads to a generalization of the first Kronecker Limit Formula. For example, Goldstein [3] finds a generalized Eisenstein series of the first Kronecker Limit Formula by looking at the Eisenstein series defined at a cusp.

Specifically, we want to look at the Fourier expansion of $E_{\chi_1,\chi_2}$ at $s = 1$. It is more convenient to consider the “completed” Eisenstein series defined by

$$E_{\chi_1,\chi_2}^*(z,s) := \frac{(q_2/\pi)^s}{i \cdot k \cdot \tau(\chi_2)} \Gamma(s + \frac{k}{2}) L(2s, \chi_1\chi_2) E_{\chi_1,\chi_2}(z,s).$$

Here $\tau$ denotes the Gauss sum given by

$$\tau(\chi) = \sum_{n=0}^{q-1} \chi(n)e^{2\pi i n/q},$$

for $\chi$ modulo $q$.

The Fourier expansion for $E_{\chi_1,\chi_2}^*$ is conveniently stated by Young [7] (see also Huxley [5]). When $q_1, q_2 \neq 1$, the Fourier expansion simplifies as

$$E_{\chi_1,\chi_2}^*(z,s) = 2\sqrt{y} \sum_{n \neq 0} \lambda_{\chi_1,\chi_2}(n,s)e(nx)K_{s-\frac{1}{2}}(2\pi|n|y),$$

where $\lambda_{\chi_1,\chi_2}$ are the Gauss sums of $\chi_1, \chi_2$.
where
\[
\lambda_{\chi_1, \chi_2}(n, s) = \chi_2(\text{sgn}(n)) \sum_{ab = |n|} \chi_1(a) \overline{\chi_2(b)} \left( \frac{b}{a} \right)^{s-\frac{1}{2}},
\]
and \(e(x) = e^{2\pi ix}\).

When \(s = 1\), the Fourier expansion simplifies as
\[
E^*_{\chi_1, \chi_2}(z, 1) = f_{\chi_1, \chi_2}(z) + \chi_2(-1)\overline{f_{\chi_1, \chi_2}(z)},
\]
where
\[
f_{\chi_1, \chi_2}(z) = \sum_{n>0} \frac{e(nz)}{\sqrt{n}} \sum_{ab=n} \chi_1(a) \overline{\chi_2(b)} \left( \frac{b}{a} \right)^{\frac{1}{2}}.
\]

Our generalized Dedekind sum arises from studying the transformation properties of \(f_{\chi_1, \chi_2}\) on \(\Gamma_0(q_1q_2)\). Other authors, such as Nagasaka \([6]\) and Goldstein and Razar \([4]\), have studied functions similar to \(f_{\chi_1, \chi_2}\) to arrive at a generalized Dedekind sum. The process we use is unique because our function \(f_{\chi_1, \chi_2}\) arises naturally from the Fourier expansion of the Eisenstein series. To obtain our Dedekind sum, we study the function \(\phi_{\chi_1, \chi_2}\), defined for \(\gamma \in \Gamma_0(q_1q_2)\) and \(z \in \mathbb{H}\) as
\[
\phi_{\chi_1, \chi_2}(\gamma, z) := f_{\chi_1, \chi_2}(\gamma z) - \psi(\gamma) f_{\chi_1, \chi_2}(z).
\]

In Lemma 2.1, we show that \(\phi_{\chi_1, \chi_2}\) is independent of \(z\).

Let \(B_1\) denote the first Bernoulli function given by
\[
B_1(x) = \begin{cases} 
  x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\
  0 & \text{if } x \in \mathbb{Z}.
\end{cases}
\]

The following theorem gives our generalized Dedekind sum in terms of the Bernoulli function.

**Theorem 1.1.** Let \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1q_2)\). Then
\[
\phi_{\chi_1, \chi_2}(\gamma) = \frac{-\pi i \chi_2(-1)}{\tau(\overline{\chi_1})} \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2(j)} \overline{\chi_1(n)} B_1 \left( \frac{j}{c} \right) B_1 \left( \frac{n}{q_1} - \frac{aj}{c} \right),
\]
where \(c = c' q_1\).

Generalized Dedekind sums arising from the Eisenstein series have been studied by Berndt \([1]\), Nagasaka \([6]\), Goldstein \([3]\), Da˘gh and Can \([2]\), and others. Berndt and others began with a different version of the Eisenstein series. Our form of the Eisenstein series is more natural and has nice arithmetical properties. Our approach is unique because our generalized Dedekind sum follows from the Fourier expansion of the Eisenstein series, and we are able to calculate the generalized Dedekind sum directly from the transformation properties of the Eisenstein series.
2 Preliminary Results

Lemma 2.1. The function $\phi_{\chi_1,\chi_2}$ is independent of $z$.

Proof. Since $E^*_{\chi_1,\chi_2}(\gamma z, 1) = \psi(\gamma)E^*_{\chi_1,\chi_2}(z, 1)$ and $E^*_{\chi_1,\chi_2}(z, 1) = f_{\chi_1,\chi_2}(z) + \chi_2(-1)f_{\chi_1,\chi_2}(\gamma z)$, it immediately follows that

$$\phi_{\chi_1,\chi_2}(\gamma, z) = -\chi_2(-1)\phi_{\chi_1,\chi_2}(\gamma, z).$$ (2.1)

Since $\phi_{\chi_1,\chi_2}$ is a holomorphic function and $\phi_{\chi_1,\chi_2}$ is an antiholomorphic function, $\phi_{\chi_1,\chi_2}$ must be constant.

From now on, we will write $\phi_{\chi_1,\chi_2}(\gamma)$ instead of $\phi_{\chi_1,\chi_2}(\gamma, z)$.

For later reference, we state a more symmetric form for $\phi_{\chi_1,\chi_2}$. Specifically, from (2.1) it follows that

$$\phi_{\chi_1,\chi_2}(\gamma) = \frac{1}{2}(\phi_{\chi_1,\chi_2}(\gamma) - \chi_2(-1)\phi_{\chi_1,\chi_2}(\gamma)).$$ (2.2)

In the following lemma, we investigate the homomorphic properties of $\phi_{\chi_1,\chi_2}$ that arise from the automorphic properties of $E^*_{\chi_1,\chi_2}$.

Lemma 2.2. Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1q_2)$. Then $\phi_{\chi_1,\chi_2}(\gamma_1\gamma_2) = \phi_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)\phi_{\chi_1,\chi_2}(\gamma_2)$.

Proof. Since $\psi$ is multiplicative,

$$\phi_{\chi_1,\chi_2}(\gamma_1\gamma_2) = f_{\chi_1,\chi_2}(\gamma_1\gamma_2z) - \psi(\gamma_1\gamma_2)f_{\chi_1,\chi_2}(z)$$
$$= f_{\chi_1,\chi_2}(\gamma_1\gamma_2z) - \psi(\gamma_1)\psi(\gamma_2)f_{\chi_1,\chi_2}(z)$$
$$= f_{\chi_1,\chi_2}(\gamma_1\gamma_2z) - \psi(\gamma_1)f_{\chi_1,\chi_2}(\gamma_2z) + \psi(\gamma_1)f_{\chi_1,\chi_2}(\gamma_2z) - \psi(\gamma_1)\psi(\gamma_2)f_{\chi_1,\chi_2}(z)$$
$$= \phi_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)\phi_{\chi_1,\chi_2}(\gamma_2).$$ □

3 The Generalized Dedekind Sum

Our main goal is to find a finite sum formula for $\phi_{\chi_1,\chi_2}$, which will give us our generalized Dedekind sum. Our process loosely follows the methodology of Goldstein [3]. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1q_2)$, and let $z = \frac{-d}{c} + \frac{i}{c^2u} \in \mathbb{H}$ for some $u \in \mathbb{R}$, $u \neq 0$. Then $\gamma z = \frac{a}{c} + iu$.

Since $\phi_{\chi_1,\chi_2}$ is independent of $z$,

$$\phi_{\chi_1,\chi_2}(\gamma) = \lim_{u \to 0^+} \left( f_{\chi_1,\chi_2} \left( \frac{a}{c} + iu \right) - \psi(\gamma) f_{\chi_1,\chi_2} \left( \frac{-d}{c} + \frac{i}{c^2u} \right) \right).$$

From the Fourier expansion of $E^*_{\chi_1,\chi_2}$, it is clear that

$$\lim_{u \to 0^+} f_{\chi_1,\chi_2} \left( \frac{-d}{c} + \frac{i}{c^2u} \right) = 0.$$ Thus,

$$\phi_{\chi_1,\chi_2}(\gamma) = \lim_{u \to 0^+} f_{\chi_1,\chi_2} \left( \frac{a}{c} + iu \right).$$ (3.1)
To evaluate this limit, we begin simplifying $f_{\chi_1,\chi_2}$ as

$$f_{\chi_1,\chi_2}(z) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\chi_1(l)\chi_2(k)}{l} e(klz). \quad (3.2)$$

Then

$$f_{\chi_1,\chi_2}(z) = \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \theta_{\chi_2}(z,l), \quad (3.3)$$

where

$$\theta_{\chi}(z,l) := \sum_{k=1}^{\infty} \chi(k)e(ckz).$$

for $\chi$ modulo $q$.

**Lemma 3.1.** Let $\chi$ modulo $q$ be a primitive character. Let $a, c, l \in \mathbb{Z}$ with $c \geq 1$, $c \equiv 0 \pmod{q}$, $(a,c) = 1$, and $l \not\equiv 0 \pmod{cq}$. Then

$$\lim_{u \to 0^+} \theta_{\chi} \left( \frac{a}{c} + iu, l \right) = \chi(-1) \sum_{j \pmod{c}} \chi(j) B_1 \left( \frac{j}{c} \right) e \left( -\frac{alj}{c} \right),$$

where $B_1$ is the first Bernoulli function given by

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\setminus\mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

**Proof.** One can write $\chi$ modulo $q$ in terms of additive characters using the formula

$$\chi(k) = \frac{1}{\tau(\chi)} \sum_{r \pmod{q}} \chi(r)e \left( \frac{rk}{q} \right). \quad (3.4)$$

Applying this, we get

$$\theta_{\chi}(z,l) = \frac{1}{\tau(\chi)} \sum_{k=1}^{\infty} \sum_{r \pmod{q}} \chi(r)e \left( klz + \frac{rk}{q} \right). \quad (3.5)$$

When $z = \frac{a}{c} + iu$,

$$\theta_{\chi} \left( \frac{a}{c} + iu, l \right) = \frac{1}{\tau(\chi)} \sum_{k=1}^{\infty} \sum_{r \pmod{q}} \chi(r)e \left( \frac{akl}{c} + iukl + \frac{rk}{q} \right) \quad (3.6)$$

$$= \frac{1}{\tau(\chi)} \sum_{r \pmod{q}} \chi(r) \frac{e \left( \frac{al}{c} + iul + \frac{r}{q} \right)}{1 - e \left( \frac{al}{c} + iul + \frac{r}{q} \right)}. \quad (3.7)$$
As \( u \to 0 \), \( e(iul) \to 1 \). Thus,
\[
\lim_{u \to 0^+} \theta_x \left( \frac{a}{c} + iu, l \right) = \frac{1}{\tau(\chi)} \sum_{r \pmod{q}} \chi(r) \frac{e \left( \frac{al}{c} + \frac{r}{q} \right)}{1 - e \left( \frac{al}{c} + \frac{r}{q} \right)}.
\]

\[
= \frac{1}{\tau(\chi)} \sum_{r \pmod{q}} \chi(r) \frac{1}{1 - e \left( -\frac{al}{c} - \frac{r}{q} \right)}.
\]

If \( \eta \neq 1 \) is a \( k \)-th root of unity, we have the following identity
\[
\frac{1}{1 - \eta} = -\frac{1}{k} \sum_{j=1}^{k-1} j\eta^j,
\]
which is easily verified by multiplying each side of the equation by \( 1 - \eta \). Since \( q|c \), \( e \left( -\frac{al}{c} - \frac{r}{q} \right) \) is a \( c \)-th root of unity. Applying this identity, we get
\[
\lim_{u \to 0^+} \theta_x \left( \frac{a}{c} + iu, l \right) = \frac{1}{\tau(\chi)} \sum_{r \pmod{q}} \chi(r) \sum_{j=1}^{c-1} j e \left( -\frac{alj}{c} - \frac{rj}{q} \right).
\]

By (3.4), this gives us
\[
\lim_{u \to 0^+} \theta_x \left( \frac{a}{c} + iu, l \right) = \frac{\chi(-1)}{\tau(\chi)} \sum_{j=1}^{c-1} j e \left( -\frac{alj}{c} \right) \tau(\chi) \overline{\chi}(j)
\]
\[
= \chi(-1) \sum_{j=1}^{c-1} \overline{\chi}(j) e \left( -\frac{alj}{c} \right).
\]

Now we want to express this in terms of \( B_1 \). We have
\[
\lim_{u \to 0^+} \theta_x \left( \frac{a}{c} + iu, l \right) = \chi(-1) \sum_{j \pmod{c}} \overline{\chi}(j) \left( \frac{j}{c} - \left[ \frac{j}{c} \right] - \frac{1}{2} + \frac{1}{2} \right) e \left( -\frac{alj}{c} \right).
\]

We have
\[
\overline{\chi}(j) \left( \frac{j}{c} - \left[ \frac{j}{c} \right] - \frac{1}{2} \right) = B_1 \left( \frac{j}{c} \right)
\]
since \( \overline{\chi}(j) = 0 \) when \( \frac{j}{c} \in \mathbb{Z} \), so the expression simplifies as
\[
\lim_{u \to 0^+} \theta_x \left( \frac{a}{c} + iu, l \right) = \left[ \chi(-1) \sum_{j \pmod{c}} \overline{\chi}(j) B_1 \left( \frac{j}{c} \right) e \left( -\frac{alj}{c} \right) \right] + \left[ \frac{\chi(-1)}{2} \sum_{j \pmod{c}} \overline{\chi}(j) e \left( -\frac{alj}{c} \right) \right].
In the second bracketed term in the line above equals zero. To see this, write \( j = A + qB \) where \( A \pmod{q} \) and \( B \pmod{c/q} \). Then

\[
\sum_{j \pmod{c}} \chi(j)e\left(\frac{-alj}{c}\right) = \sum_{A \pmod{q}} \chi(A)e\left(\frac{-alA}{c}\right) \sum_{B \pmod{c/q}} e\left(\frac{-alB}{c/q}\right).
\]

Since \( \frac{c}{q} \not| al \), the sum over \( B \) equals 0. Thus,

\[
\lim_{u \to 0^+} \theta_\chi\left(\frac{a}{c} + iu, l\right) = \chi(-1) \sum_{j \pmod{c}} \chi(j)B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right).
\]

Now we simplify \( \phi_{\chi_1, \chi_2} \) further using the function \( \theta_\chi \).

**Lemma 3.2.** Let \( \gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_0(q_1q_2) \). Then

\[
\phi_{\chi_1, \chi_2}(\gamma) = \chi_2(-1) \sum_{l=1}^\infty \frac{\chi_1(l)}{l} \sum_{j \pmod{c}} \chi_2(j)B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right).
\]

**Proof.** We apply Lemma 3.1 to 3.3.

\[
\phi_{\chi_1, \chi_2}(\gamma) = \lim_{u \to 0} \sum_{l=1}^\infty \frac{\chi_1(l)}{l} \theta_{\chi_2}\left(\frac{a}{c} + iu, l\right)
= \sum_{l=1}^\infty \frac{\chi_1(l)}{l} \lim_{u \to 0} \theta_{\chi_2}\left(\frac{a}{c} + iu, l\right)
= \sum_{l=1}^\infty \frac{\chi_1(l)}{l} \chi_2(-1) \sum_{j \pmod{c}} B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right). \quad \square
\]

**Remark.** The following theorem relies on the generalized Bernoulli function, which is stated by Berndt [1]. One may easily simplify Berndt’s formulae to get the equivalent form

\[
B_{1, \chi}(x) = \frac{-\tau(\chi)}{2\pi i} \sum_{\substack{l \in \mathbb{Z} \setminus 0}} \frac{\chi(l)}{l} e\left(\frac{lx}{q}\right)
\]

(3.8)

for \( \chi \) modulo \( q \). We prefer using this form because it is more symmetric and does not depend on the parity of \( \chi \).

**Proof of Theorem 1.1.** By applying Lemma 3.2 to (2.2), we get the simplification below.

\[
\phi_{\chi_1, \chi_2}(\gamma) = \frac{\chi_2(-1)}{2} \sum_{l=1}^\infty \frac{\chi_1(l)}{l} \sum_{j \pmod{c}} \chi_2(j)B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right)
- \chi_2(-1) \frac{\chi_2(-1)}{2} \sum_{l=1}^\infty \frac{\chi_1(l)}{l} \sum_{j \pmod{c}} \chi_2(j)B_1\left(\frac{j}{c}\right) e\left(\frac{alj}{c}\right).
\]
Using the change of variables $l \rightarrow -l$ and the fact that $\chi_1(-1)\chi_2(-1) = 1$, this simplifies as

$$
\phi_{\chi_1,\chi_2}(\gamma) = \frac{\chi_2(-1)}{2} \sum_{l \neq 0} \frac{\chi_1(l)}{l} \sum_{j \pmod{c}} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) e\left(-\frac{alj}{c}\right)
$$

$$
= \frac{\chi_2(-1)}{2} \sum_{j \pmod{c}} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) \sum_{l \neq 0} \frac{\chi_1(l)}{l} e\left(-\frac{alj}{c}\right).
$$

Letting $c = c'q_1$ and substituting $B_1;\chi_1$ into this expression, we get

$$
\phi_{\chi_1,\chi_2}(\gamma) = \frac{-\pi i \chi_2(-1)}{\tau(\chi_1)} \sum_{j \pmod{c}} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) B_1;\chi_1\left(-\frac{aj}{c'}\right).
$$

This sum may also be written as a finite expression using the following transformation formula by Berndt [1],

$$
B_1;\chi(x) = \sum_{n=1}^{q-1} \overline{\chi}(n) B_1\left(\frac{x+n}{q}\right),
$$

for $\chi$ modulo $q$.

Substituting this in, we get

$$
\phi_{\chi_1,\chi_2}(\gamma) = \frac{-\pi i \chi_2(-1)}{\tau(\chi_1)} \sum_{j \pmod{c}} \sum_{n \pmod{q_1}} \overline{\chi_2}(j) \overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} - \frac{aj}{c}\right).
$$
References


