Neural Bonanza III: The Final Bonanza Pt. 1

Brianna Gambacini

University of Connecticut

REU 2019
Texas A&M University
Joint work with Sam Macdonald (Willamette)

July 22, 2019
Outline

• Biological motivation
• Definitions
• Disproving conjectures
• Main question
• Future research
Biological Motivation

- Place cells in hippocampus
- Encode data
- Maps environment
- Convex place fields
Biological Motivation

- Place cells in hippocampus
- Encode data
- Maps environment
- Convex place fields

Relation to Mathematics

Can we find criteria to classify neural codes as convex given only the structure of the code?
Important Definitions

Open/Closed Convex Codes

A code $C \subset 2^n$ is open (or closed) convex if there exist open (or closed) convex subsets $U_1, U_2, \ldots, U_n \subseteq \mathbb{R}^d$, for some $d$, that generate the code.
A code $C$ is \textit{3-sparse} if no codeword is longer than 3 neurons.

Let $C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\}$
Important Definitions

3-Sparse
A code $C$ is 3-sparse if no codeword is longer than 3 neurons.

Let $C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\}$

Facet
A codeword $\sigma \in C$ is a facet if it is a maximal element of $C$ with respect to inclusion, that is, $\sigma \not\subseteq \alpha$ for all $\alpha \in C$ such that $\alpha \neq \sigma$. 
**Important Definitions**

### 3-Sparse

A code $C$ is *3-sparse* if no codeword is longer than 3 neurons.

Let $C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\}$

### Facet

A codeword $\sigma \in C$ is a *facet* if it is a maximal element of $C$ with respect to inclusion, that is, $\sigma \not\subseteq \alpha$ for all $\alpha \in C$ such that $\alpha \neq \sigma$.

Here, our facets are $\{123, 124, 34\}$
Important Definitions

\[ C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\} \]

**Max-Intersection-Complete**

A code \( C \) is *max-intersection complete* if all the intersections of its facets are in \( C \).
Important Definitions

\[ C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\} \]

Max-Intersection-Complete

A code \( C \) is\(^{\text{max-intersection complete}}\) if all the intersections of its facets are in \( C \).

- Facets: \( \{123, 124, 34\} \)
Important Definitions

$$C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\}$$

Max-Intersection-Complete

A code $C$ is *max-intersection complete* if all the intersections of its facets are in $C$.

- Facets: $\{123, 124, 34\}$
- Intersections: $12 = 123 \cap 124$, $3 = 123 \cap 34$, $4 = 124 \cap 34$
$C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\}$

Max-Intersection-Complete

A code $C$ is *max-intersection complete* if all the intersections of its facets are in $C$.

- Facets: $\{123, 124, 34\}$
- Intersections: $12 = 123 \cap 124$, $3 = 123 \cap 34$, $4 = 124 \cap 34$
- $\{12, 3, 4\} \subseteq C$
Important Definitions

$$C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\}$$

Max-Intersection-Complete

A code $C$ is \textit{max-intersection complete} if all the intersections of its facets are in $C$.

- Facets: $\{123, 124, 34\}$
- Intersections: $12 = 123 \cap 124$, $3 = 123 \cap 34$, $4 = 124 \cap 34$
- $\{12, 3, 4\} \subseteq C$
- So $C$ is max-intersection complete
Important Definitions

\[ C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\} \]

Simplicial Complex

We define the *simplicial complex* of a code \( C \) as:

\[ \Delta(C) := \{\sigma \subseteq [n] : \sigma \subseteq \alpha \text{ for some } \alpha \in C\} \]
Important Definitions

\[ C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\} \]

**Simplicial Complex**

We define the *simplicial complex* of a code \( C \) as:

\[ \Delta(C) := \{\sigma \subseteq [n] : \sigma \subseteq \alpha \text{ for some } \alpha \in C\} \]

\[ \Delta(C) = \{123, 124, 34, 12, 13, 14, 23, 24, 34, 1, 2, 3, 4, \emptyset\} \]
Important Definitions

$C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\}$

**Simplicial Complex**

We define the *simplicial complex* of a code $C$ as:

$$\Delta(C) := \{\sigma \subseteq [n] : \sigma \subseteq \alpha \text{ for some } \alpha \in C\}$$

$$\Delta(C) = \{123, 124, 34, 12, 13, 14, 23, 24, 34, 1, 2, 3, 4, \emptyset\}$$

**Link**

For a simplicial complex $\Delta$ and some $\sigma \in \Delta$, the *link* of $\sigma$ is defined as:

$$\text{Lk}_{\sigma}(\Delta) := \{\tau \subseteq [n] \setminus \sigma : \sigma \cup \tau \in \Delta\}$$
Important Definitions

\[ C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\} \]

**Simplicial Complex**

We define the *simplicial complex* of a code \( C \) as:

\[ \Delta(C) := \{\sigma \subseteq [n] : \sigma \subseteq \alpha \text{ for some } \alpha \in C\} \]

\[ \Delta(C) = \{123, 124, 34, 12, 13, 14, 23, 24, 34, 1, 2, 3, 4, \emptyset\} \]

**Link**

For a simplicial complex \( \Delta \) and some \( \sigma \in \Delta \), the *link* of \( \sigma \) is defined as:

\[ \text{Lk}_\sigma(\Delta) := \{\tau \subseteq [n] \setminus \sigma : \sigma \cup \tau \in \Delta\} \]

\[ \text{Lk}_{\{3\}}(\Delta(C)) = \{12, 4, 1, 2, \emptyset\} \]
Important Definitions

\[ C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\} \]

**Mandatory**

A word \( \sigma \in \Delta(C) \) is *mandatory* if \( \text{Lk}_\sigma(\Delta(C)) \) is not contractible. Similarly, \( \sigma \) is *non-mandatory* if \( \text{Lk}_\sigma(\Delta(C)) \) is contractible.
Important Definitions

\( C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\} \)

**Mandatory**

A word \( \sigma \in \Delta(C) \) is *mandatory* if \( \text{Lk}_\sigma(\Delta(C)) \) is not contractible. Similarly, \( \sigma \) is *non-mandatory* if \( \text{Lk}_\sigma(\Delta(C)) \) is contractible.

\[
\text{Lk}_{\{3\}}(\Delta(C)) = \{12, 4, 1, 2, \emptyset\}
\]

\[
\text{Lk}_{\{1\}}(\Delta(C)) = \{23, 24, 2, 3, 4, \emptyset\}
\]
Important Definitions

\[ C = \{123, 124, 12, 13, 34, 1, 3, 4, \emptyset\} \]

Mandatory

A word \( \sigma \in \Delta(C) \) is \textit{mandatory} if \( \text{Lk}_\sigma(\Delta(C)) \) is not contractible. Similarly, \( \sigma \) is \textit{non-mandatory} if \( \text{Lk}_\sigma(\Delta(C)) \) is contractible.

\[ \text{Lk}_{\{3\}}(\Delta(C)) = \{12, 4, 1, 2, \emptyset\} \quad \text{Lk}_{\{1\}}(\Delta(C)) = \{23, 24, 2, 3, 4, \emptyset\} \]

Locally Good

A code is \textit{locally good} if it contains all of its mandatory codewords.
Disproven Conjectures: Goldrup and Phillipson

Conjecture (Goldrup and Phillipson 2014)

Let $C$ be a code that is open convex, not max intersection-complete, and has at least two non-mandatory codewords. Suppose $C$ has at least 3 facets $M_1, M_2, M_3$, and there is $\sigma \in C$ such that $\sigma \subset M_1$ and $\sigma \cap M_2 \notin C$. Then $C$ is not a closed convex code.

$$C = \{135, 123, 236, 124, 12, 13, 14, 23, 24, 1, 2, \emptyset\}$$
Goldrup and Phillipson Conjecture

- Open convex

\[ C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\} \]
Goldrup and Phillipson Conjecture

- Open convex
- Not max-\(\cap\)-complete

\[C = \{1, 13, 14, \underline{135}, \underline{123}, 12, \underline{124}, \underline{236}, 23, 24, 2, \emptyset\}\]
Goldrup and Phillipson Conjecture

- Open convex
- Not max-∩-complete
  - $135 \cap 236 = 3 \notin C$

$C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}$
Goldrup and Phillipson Conjecture

- Open convex
- Not max-$\cap$-complete
  - $135 \cap 236 = 3 \notin C$
- $\geq 2$ non-mandatory words

$C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}$
Goldrup and Phillipson Conjecture

- Open convex
- Not max-∩-complete
  - $135 \cap 236 = 3 \notin C$
- $\geq 2$ non-mandatory words
  - $\text{Lk}_3(\Delta) = \{15, 12, 26, \emptyset\}$
  - $\text{Lk}_4(\Delta) = \{12, \emptyset\}$
- $C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}$
Goldrup and Phillipson Conjecture

- Open convex
- Not max-$\cap$-complete
  - $\{135 \cap 236 = 3 \notin C$
- $\geq 2$ non-mandatory words
  - $\text{Lk}_3(\Delta) = \{15, 12, 26, \emptyset\}$
  - $\text{Lk}_4(\Delta) = \{12, \emptyset\}$
- $\geq 3$ facets $M_1, M_2, M_3$

\[C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}\]
Goldrup and Phillipson Conjecture

- Open convex
- Not max-∩-complete
  - \(135 \cap 236 = 3 \notin C\)
- \(\geq 2\) non-mandatory words
  - \(\text{Lk}_3(\Delta) = \{15, 12, 26, \emptyset\}\)
  - \(\text{Lk}_4(\Delta) = \{12, \emptyset\}\)
- \(\geq 3\) facets \(M_1, M_2, M_3\)
  - \(M_1 = 123, M_2 = 236, M_3 = 135\)

\[C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}\]
Goldrup and Phillipson Conjecture

- Open convex
- Not max-$\cap$-complete
  - $135 \cap 236 = 3 \notin C$
- $\geq 2$ non-mandatory words
  - $\text{Lk}_3(\Delta) = \{15, 12, 26, \emptyset\}$
  - $\text{Lk}_4(\Delta) = \{12, \emptyset\}$
- $\geq 3$ facets $M_1, M_2, M_3$
  - $M_1 = 123$, $M_2 = 236$, $M_3 = 135$
- $\sigma \in C$ such that:
  - $\sigma \subset M_1$. 

$C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}$
Goldrup and Phillipson Conjecture

- Open convex
- Not max-∩-complete
  - \(135 \cap 236 = 3 \notin C\)
- \(\geq 2\) non-mandatory words
  - \(Lk\{3\}(\Delta) = \{15, 12, 26, \emptyset\}\)
  - \(Lk\{4\}(\Delta) = \{12, \emptyset\}\)
- \(\geq 3\) facets \(M_1, M_2, M_3\)
  - \(M_1 = 123, M_2 = 236, M_3 = 135\)
- \(\sigma \in C\) such that:
  - \(\sigma \subset M_1\). Let \(\sigma = 13. 13 \subset 123\)

\[C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}\]
Goldrup and Phillipson Conjecture

- Open convex
- Not max-$\cap$-complete
  - $135 \cap 236 = 3 \notin C$
- $\geq 2$ non-mandatory words
  - $\text{Lk}_3(\Delta) = \{15, 12, 26, \emptyset\}$
  - $\text{Lk}_4(\Delta) = \{12, \emptyset\}$
- $\geq 3$ facets $M_1$, $M_2$, $M_3$
  - $M_1 = 123$, $M_2 = 236$, $M_3 = 135$
- $\sigma \in C$ such that:
  - $\sigma \subset M_1$. Let $\sigma = 13$. $13 \subset 123$
  - $\sigma \cap M_2 \notin C$.

$C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}$
Goldrup and Phillipson Conjecture

- Open convex
- Not max-∩-complete
  - $135 \cap 236 = 3 \not\in C$
- $\geq 2$ non-mandatory words
  - $\text{Lk}_{\{3\}}(\Delta) = \{15, 12, 26, \emptyset\}$
  - $\text{Lk}_{\{4\}}(\Delta) = \{12, \emptyset\}$
- $\geq 3$ facets $M_1, M_2, M_3$
  - $M_1 = 123, M_2 = 236, M_3 = 135$
- $\sigma \in C$ such that:
  - $\sigma \subset M_1$. Let $\sigma = 13$. $13 \subset 123$
  - $\sigma \cap M_2 \notin C$. $13 \cap 236 = 3 \not\in C$

$C = \{1, 13, 14, 135, 123, 12, 124, 236, 23, 24, 2, \emptyset\}$
Goldrup and Phillipson Conjecture

- Open convex
- Not max-$\cap$-complete
  - $135 \cap 236 = 3 \notin C$
- $\geq 2$ non-mandatory words
  - $\text{Lk}_3(\Delta) = \{15, 12, 26, \emptyset\}$
  - $\text{Lk}_4(\Delta) = \{12, \emptyset\}$
- $\geq 3$ facets $M_1, M_2, M_3$
  - $M_1 = 123, M_2 = 236, M_3 = 135$
- $\sigma \in C$ such that:
  - $\sigma \subset M_1$. Let $\sigma = 13$. $13 \subset 123$
  - $\sigma \cap M_2 \notin C$. $13 \cap 236 = 3 \notin C$

Then the Conjecture says $C$ is not closed convex...
Goldrup and Phillipson Conjecture

- Open convex
- Not max-$\cap$-complete
  - $135 \cap 236 = 3 \notin C$
- $\geq 2$ non-mandatory words
  - $\text{Lk}_\{3\}(\Delta) = \{15, 12, 26, \emptyset\}$
  - $\text{Lk}_\{4\}(\Delta) = \{12, \emptyset\}$
- $\geq 3$ facets $M_1, M_2, M_3$
  - $M_1 = 123, M_2 = 236, M_3 = 135$
- $\sigma \in C$ such that:
  - $\sigma \subset M_1$. Let $\sigma = 13. 13 \subset 123$
  - $\sigma \cap M_2 \notin C. 13 \cap 236 = 3 \notin C$

Then the Conjecture says $C$ is not closed convex... but this is false!
Main Question

What we already know:

- Convex $\Rightarrow$ locally good
Main Question

What we already know:

- Convex $\Rightarrow$ locally good
- 2-sparse, locally good $\Rightarrow$ convex

Conjecture 1
If a 3-sparse neural code is locally good, then it must be closed convex.

Conjecture 2
If a 3-sparse neural code is locally good, then it must be open convex.
What we already know:

- Convex $\Rightarrow$ locally good
- 2-sparse, locally good $\Rightarrow$ convex
- $4^+$-sparse, locally good $\not\Rightarrow$ convex
Main Question

What we already know:

- Convex $\Rightarrow$ locally good
- 2-sparse, locally good $\Rightarrow$ convex
- $4^+$-sparse, locally good $\not\Rightarrow$ convex
- But what about 3-sparse codes?

Conjecture 1
If a 3-sparse neural code is locally good, then it must be closed convex.

Conjecture 2
If a 3-sparse neural code is locally good, then it must be open convex.
Main Question

What we already know:

- Convex $\Rightarrow$ locally good
- 2-sparse, locally good $\Rightarrow$ convex
- $4^+$-sparse, locally good $\not\Rightarrow$ convex
- But what about 3-sparse codes?

Conjecture 1

If a 3-sparse neural code is locally good, then it must be **closed** convex.
What we already know:

- Convex ⇒ locally good
- 2-sparse, locally good ⇒ convex
- $4^+$-sparse, locally good $\not\Rightarrow$ convex
- But what about 3-sparse codes?

Conjecture 1
If a 3-sparse neural code is locally good, then it must be **closed** convex.

Conjecture 2
If a 3-sparse neural code is locally good, then it must be **open** convex.
Conjecture 1

If a 3-sparse neural code is locally good, then it must be **closed** convex.

\[ C = \{123, 124, 235, 12, 14, 23, 35, 45, 4, 5, \emptyset\} \]

Recall: Open convex \( \Rightarrow \) locally good
Closed Convex
Closed Convex
Closed Convex
Closed Convex
Closed Convex
Closed Convex
Closed Convex
Theorem 4.3 (G. and Macdonald)

Let $C$ be a neural code on $n$ neurons with a closed convex cover $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$ that is fully dimensional. For $\sigma \subset [n]$, we define $U_\sigma = \bigcap_{i \in \sigma} U_i$. If there does not exist an $\alpha \in C$ such that $U_\alpha$ consists of a set that cannot be drawn in $\mathbb{R}^{d-1}$ or higher, then $C$ is open convex.
Theorem 4.3 (G. and Macdonald)

Let $C$ be a neural code on $n$ neurons with a closed convex cover $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$ that is fully dimensional. For $\sigma \subset [n]$, we define $U_\sigma = \cap_{i \in \sigma} U_i$. If there does not exist an $\alpha \in C$ such that $U_\alpha$ consists of a set that cannot be drawn in $\mathbb{R}^{d-1}$ or higher, then $C$ is open convex.
Theorem 4.3 (G. and Macdonald)

Let $C$ be a neural code on $n$ neurons with a closed convex cover $U = \{U_i\}_{i=1}^{n}$ in $\mathbb{R}^d$ that is fully dimensional. For $\sigma \subset [n]$, we define $U_\sigma = \bigcap_{i \in \sigma} U_i$. If there does not exist an $\alpha \in C$ such that $U_\alpha$ consists of a set that cannot be drawn in $\mathbb{R}^{d-1}$ or higher, then $C$ is open convex.
Theorem 4.3 (G. and Macdonald)

Let $C$ be a neural code on $n$ neurons with a closed convex cover $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$ that is fully dimensional. For $\sigma \subset [n]$, we define $U_\sigma = \bigcap_{i \in \sigma} U_i$. If there does not exist an $\alpha \in C$ such that $U_\alpha$ consists of a set that cannot be drawn in $\mathbb{R}^{d-1}$ or higher, then $C$ is open convex.
Theorem 4.3 (G. and Macdonald)

Let $C$ be a neural code on $n$ neurons with a closed convex cover $U = \{U_i\}_{i=1}^{n}$ in $\mathbb{R}^d$ that is fully dimensional. For $\sigma \subset [n]$, we define $U_\sigma = \cap_{i \in \sigma} U_i$. If there does not exist an $\alpha \in C$ such that $U_\alpha$ consists of a set that cannot be drawn in $\mathbb{R}^{d-1}$ or higher, then $C$ is open convex.
Lemma 4.4 (G. and Macdonald)

Let $C$ be a neural code on $n$ neurons with a closed convex cover $U = \{U_i\}_{i=1}^{n}$ in $\mathbb{R}^d$. If there exists a $U_\alpha$ that can only be expressed in $\mathbb{R}^{d-2}$ or below and is the intersection of exactly two sets in $U$, then $C$ is open convex.
Lemma 4.4 (G. and Macdonald)

Let $C$ be a neural code on $n$ neurons with a closed convex cover $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$. If there exists a $U_\alpha$ that can only be expressed in $\mathbb{R}^{d-2}$ or below and is the intersection of exactly two sets in $U$, then $C$ is open convex.
Lemma 4.4 (G. and Macdonald)

Let $C$ be a neural code on $n$ neurons with a closed convex cover $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$. If there exists a $U_\alpha$ that can only be expressed in $\mathbb{R}^{d-2}$ or below and is the intersection of exactly two sets in $U$, then $C$ is open convex.
Possible Future Research

Conjecture

If $C$ is a 3-sparse, locally good neural code on $n$ neurons that is closed convex, then $C$ is also open convex.
Conjecture
If $C$ is a 3-sparse, locally good neural code on $n$ neurons that is closed convex, then $C$ is also open convex.

Next Up
Find and define other criteria for open convexity that does not depend on closed convexity.
Thank you for listening!
Special thanks to Dr. Anne Shiu, Nida Obatake, Thomas Yahl, and the National Science Foundation.
References

