Neural Bonanza III

The Final Bonanza, Pt. II

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**Definition**
A code $C$ is **max-intersection complete** if all the intersections of its facets are in $C$. If a code does not contain all of its facets’ intersections then it is **max-intersection incomplete**.

**Definition**
For a neural code $C$ on $n$ vertices, the **simplicial complex** $\Delta(C)$ is a subset of $2^[[n]]$ that is closed under taking subsets, where $[n] := \{1, 2, \ldots, n\}$ is the population of neurons. More specifically:

$$\Delta(C) := \{\sigma \subseteq [n] : \sigma \subseteq \alpha \text{ for some } \alpha \in C\}.$$  

**Definition**
Let $\Delta$ be a simplicial complex on $n$ vertices and $\sigma \in \Delta$. Then the **link** of $\sigma$ in $\Delta$ is:

$$\text{Lk}_\sigma(\Delta) := \{\tau \subseteq [n] \setminus \sigma : \sigma \cup \tau \in \Delta\}.$$
Theorem

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**Proof.**

Let $C$ be a locally good neural code on $n$ neurons with distinct facets $M_1, M_2, \ldots, M_n$ such that no intersection of facets contains more than one neuron.
Complete to the Max

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Suppose by way of contradiction that $C$ is max intersection incomplete. Thus there must exist some neuron $\sigma \notin C$ and $M_i, M_j \in C$ such that $M_i \cap M_j = \sigma$. 
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As $M_i$ and $M_j$ are distinct, there must exist $\alpha, \beta$ such that $M_i = \sigma \alpha$ and $M_j = \sigma \beta$. 
Proof.

Consider $\text{Lk}_\sigma(\Delta)$. Recall that $\text{Lk}_\sigma(\Delta) := \{\tau \subseteq [n] \backslash \sigma : \sigma \cup \tau \in \Delta\}$.
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As $\sigma \cup \alpha$ and $\sigma \cup \beta \in \Delta(C)$, it must be the case that $\alpha, \beta \in Lk_\sigma(\Delta)$.
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Consider $\text{Lk}_\sigma(\Delta)$. Recall that $\text{Lk}_\sigma(\Delta) := \{\tau \subseteq [n] \setminus \sigma : \sigma \cup \tau \in \Delta\}$.

As $\sigma \cup \alpha$ and $\sigma \cup \beta \in \Delta(\mathcal{C})$, it must be the case that $\alpha, \beta \in \text{Lk}_\sigma(\Delta)$.

Thus, as it stands, $\text{Lk}_\sigma(\Delta)$ is the following:

\[
\begin{array}{c c}
\alpha & \beta \\
\bullet & \bullet
\end{array}
\]

which is not contractible. There are three ways to make this link contactable, and we will show how each leads to a contradiction.
**Proof.**

This would introduce $\sigma \alpha \beta$ to the code. This is either a facet of $C$ or a subset of some facet in $C$. Either way, the intersection of this facet with $M_i$ is $\sigma \alpha$, a contradiction.
Case II: There Exists Exactly One $\lambda$

Proof.

This would introduce $\sigma\alpha\lambda$ and $\sigma\beta\lambda$ to the code. These codewords are either facets of $C$ or subsets of other facets in $C$. Either way, the intersection of these facets is $\sigma\lambda$, a contradiction.
Case III: There Exists a Finite Number of $\lambda$s

Proof.

This would introduce quite a few things to the code. However, just focusing on $\alpha, \lambda_1, \lambda_2$, we see that both $\sigma \alpha \lambda_1$ and $\sigma \lambda_1 \lambda_2$ are in the code, meaning this case also results in contradiction. Thus, $C$ must be max intersection complete. $\square$
Theorem

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Then there must exist some neuron $\sigma \notin C$ and $M_i, M_j \in C$ such that $M_i \cap M_j = \sigma$. 
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The facets $M_i$ and $M_j$ must be of at least length two to remain distinct, given their shared $\sigma$. 
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The facets $M_i$ and $M_j$ must be of at least length two to remain distinct, given their shared $\sigma$.

However, if $M_i$ and $M_j$ were of length two, then $\sigma$ would be a mandatory codeword.
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However, this could not be the entire code, as this would make $\sigma$ a mandatory codeword. As $C$ is 3-sparse, there must exist some other facet $M_k \in C$ such that $M_i \cap M_j \cap M_k = \sigma$. 
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To remain distinct from $M_i$ and $M_j$, $M_k$ must contain some neuron $\tau \notin M_i, M_j$. However, if both $M_i \cap M_k = \sigma$ and $M_j \cap M_k = \sigma$, then there would exist a local obstruction at $\sigma$. 

Thus, without loss of generality, there must exist some $\alpha$ such that $M_i \cap M_k = \sigma^\alpha$. Therefore, as $C$ is 3-sparse we know that $M_k = \sigma^\alpha$, completing the proof. □
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Thus, without loss of generality, there must exist some $\alpha$ such that $M_i \cap M_k = \sigma \alpha$. Therefore, as $C$ is 3-sparse we know that $M_k = \sigma \alpha \tau$, completing the proof. □
Definition

For a 3-sparse neural code, the **reduced** code of $C$, denoted $C_{\text{red}}$, is the code containing all length three codewords of $C$ and their subsets that are also in $C$. 

Example

Consider the following neural code:

\[ C = \{ \text{one.osf/two.osf/three.osf}, \text{one.osf/three.osf/four.osf}, \text{one.osf/four.osf/five.osf}, \text{one.osf/three.osf}, \text{one.osf/four.osf}, \text{two.osf/six.osf}, \text{two.osf/seven.osf}, \text{two.osf/nine.osf}, \text{three.osf/five.osf}, \text{three.osf/seven.osf}, \text{three.osf/eight.osf}, \text{four.osf/six.osf}, \text{four.osf/eight.osf}, \text{four.osf/nine.osf}, \text{five.osf/eight.osf}, \text{six.osf/seven.osf}, \text{seven.osf/nine.osf}, \text{eight.osf/nine.osf}, \text{two.osf}, \text{three.osf}, \text{four.osf}, \text{five.osf}, \text{six.osf}, \text{seven.osf}, \text{eight.osf}, \text{nine.osf}, \emptyset \} \]

$C_{\text{red}} = \{ \text{one.osf/two.osf/three.osf}, \text{one.osf/three.osf/four.osf}, \text{one.osf/four.osf/five.osf}, \text{one.osf/three.osf}, \text{one.osf/four.osf}, \emptyset \}$
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**Example**

Consider the following neural code:

$$C = \{123, 134, 145, 13, 14, 26, 27, 29, 35, 37, 38, 46, 48, 49, 58, 67, 79, 89, 2, 3, 4, 5, 6, 7, 8, 9, \emptyset\}.$$
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$$C_{\text{red}} = \{123, 134, 145, 13, 14, \emptyset \}$$
**Theorem**

Let $C$ be a $3$-sparse neural code on $n$ neurons. If there exists a closed convex cover $U = \{U_i\}_{i=1}^{n}$ in $\mathbb{R}^d$ of $C_{\text{red}}$ such that every set in $U$ can be realized as fully $\mathbb{R}^{d-1}$ or higher, then $C$ is open convex.
Theorem

Let $C$ be a 3-sparse neural code on $n$ neurons. If there exists a closed convex cover $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$ of $C_{red}$ such that every set in $U$ can be realized as fully $\mathbb{R}^{d-1}$ or higher, then $C$ is open convex.

Proof.
Let $C$ be a 3-sparse locally good neural code on $n$ neurons. Suppose that there exists some fully dimensional closed cover of $C_{red}$, denoted $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$. We will construct an open cover of $C$ using $U$. 
Proof.

\[ C = \{123, 134, 145, 13, 14, 26, 27, 29, 35, 37, 38, 46, 48, 49, 58, 67, 79, 89, 2, 3, 4, 5, 6, 7, 8, 9, \emptyset\} \]
**Step One: Intersections of Neurons in** \( C_{\text{red}} \)

**Proof.**

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**Step One: Intersections of Neurons in** $C_{red}$

**Proof.**
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Using the same epsilonic procedure as was used in Theorem 4.3, we can make this new realization fully dimensional.
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The only neurons missing from $U$ are the ones not involved in any triple-wise intersection. Let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset C$ denote the set of these neurons.
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Begin with some $\alpha_i$ such that $\alpha_i$ fires with some neuron $\beta_1 \in C_{red}$. 
**Step Two: Neurons in C but not \(C_{\text{red}}\)**

**Proof.**

The only neurons missing from \(U\) are the ones not involved in any triple-wise intersection. Let \(A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq C\) denote the set of these neurons.

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\]
**Step Two: Neurons in C but not $C_{red}$**

**Proof.**

\[ C = \{123, 134, 145, 13, 14, 26, 27, 29, 35, 37, 38, 46, 48, 49, 58, 67, 79, 89, 2, 3, 4, 5, 6, 7, 8, 9, \emptyset\} . \]
In general, either $\alpha_i \in C$ or it isn’t. If not, draw it as a subset of $\beta_1$. If so, draw it so that it overlaps with $\beta_1$. 

**Proof.**
Repeat this process for each $1 \leq i \leq n$, each time selecting a codeword that contains a neuron already existing in the realization. This provides us with a fully dimensional closed realization of $C$. 

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Thus, by Theorem 4.3, $C$ is convex. □
**Conjecture**

Let $C$ be a closed convex neural code on $n$ neurons. Let $U = \{U_i\}_{i=1}^n$ in $\mathbb{R}^d$ be an arbitrary open convex cover of $C$. If filling in the boundary of each $U_i \in U$ will always create a set that can only be realized in $\mathbb{R}^{d-2}$ or below, then $C$ is not open convex.
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Conjecture
Let $C$ be a locally good neural code on $n$ neurons. If $C$ is not open convex, then any convex realization of $C$ in $\mathbb{R}^d$ must contain a set that can only be realized in $\mathbb{R}^{d-2}$ or below.
### Conjecture

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### Conjecture

Let $C$ be a locally good neural code on $n$ neurons. If $n \leq 7$, then $C$ must be either open or closed convex.


• Sarah Ayman Goldrup and Kaitlyn Phillipson. Classification of open and closed convex codes on five neurons, 2014.