Probability of Easily Approximating Positive Reals
Roots of Trinomials

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Outline

1. Notation

2. Failure Probability vs. Exponent Ratio

3. Failure Probability vs. Variance Ratio

4. Upper Bounding Failure Probability vs. Variance Ratio
   - Small sigma: linear
   - Large sigma: $x^{-k}$
Univariate Trinomials

Let \( f(x) = c_1 x^{\alpha_0} + c_2 x^{\alpha_1} + c_3 x^{\alpha_2} \)

- \( \alpha_0 < \alpha_1 < \alpha_2 \)
- \( c_i \sim N(0, \sigma_i) \)
- generally, \( \alpha_0 = 0 \)
Spread

\[
\text{spread}(f) := \frac{\min(\alpha_1 - \alpha_0, \alpha_2 - \alpha_1)}{\alpha_2 - \alpha_0}
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- \[ \text{spread}(c_1 x^{\alpha_0} + c_2 x^{\frac{\alpha_0 + \alpha_2}{2}} + c_3 x^{\alpha_2}) = 0.5 \]
- As \( \alpha_1 \to \alpha_0 \) or \( \alpha_2 \), \( \text{spread}(f) \to 0 \)
Experimental Consideration

What is the relationship between the spread of a trinomial $f$ and its failure probability?
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**Method:**
- fix $\alpha_2$
- iterate $\alpha_1$ from $[1, \alpha_2 - 1]$
- 1,000,000 trials per ratio
- generate new random standard Gaussian coefficients each trial
Trinomial Exponent Ratio: Results I

\[ f = c_1 + c_2 x^{\alpha_1} + c_3 x^{100} \]

- 99 exponent ratios
- scipy's \texttt{curve_fit} function
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\[ h(x) = 0.61353465 + 21.87751589x - 21.86653471x^2 \]
Trinomial Exponent Ratio: Results II

\[ f = c_1 + c_2 x^{\alpha_1} + c_3 x^{100} \]
- 99 exponent ratios
- \( h(x) = 0.61353465 + 21.87751589x - 21.86653471x^2 \)

\[ f = c_1 + c_2 x^{\alpha_1} + c_3 x^{25} \]
- 24 exponent ratios
- \( h(x) = 0.70218905 + 21.39398914x - 21.38648046x^2 \)

\[ f = c_1 + c_2 x^{\alpha_1} + c_3 x^{1987} \]
- \( \alpha_1 \in [19, 1900] \)
- \( h(x) = 0.65875168 + 21.56950267x - 21.5027753x^2 \)
Trinomial Exponent Ratio: Results III

\[ f = c_1 x^{24} + c_2 x^{a_1} + c_3 x^{626} \]

- 100 exponent ratios
- x-axis \( \frac{24}{a_1} \)

\[
\begin{align*}
 h(x) &= -0.27225719 + 23.51542209x - 21.77854389x^2 
\end{align*}
\]
Experimental Hypotheses

- The graph of the failure probability as a function of trinomial spread is, roughly, a parabola or ellipse.
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- Failure probability appears to never exceed 6%.
- Failure probability also depends on variance ratios.
Experimental Consideration

What is the relationship between the failure probability of $f$, a quadratic polynomial, and $\frac{\sigma^2}{\sigma_1}$, recalling that $c_i \sim N(0, \sigma_i)$?
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What is the relationship between the failure probability of $f$, a quadratic polynomial, and $\frac{\sigma_2}{\sigma_1}$, recalling that $c_i \sim N(0, \sigma_i)$?

Method:

- 100 values of $\sigma_2$ in [0.1, 10]
- 1,000,000 trials per ratio
- generate $c_1$ and $c_3$ from standard Gaussian distributions, and $c_2$ from $N(0, \sigma_2)$ each trial
Varying the standard deviation of $c_2$:

- $\sigma_2 \in [0.1, 10]$

Figure: Quadratic $\sigma_2$ vs. Failure Probability

$$h(x) = -1.03061413 + 15.572038x^{1.0356945}e^{-1.04617418x} + 1.76374323xe^{-0.20716401x}$$
Varying the standard deviation of $c_3$:

- $\sigma_3 \in [0.1, 100]$
Varying the standard deviation of $c_3$:

$\sigma_3 \in [0.1, 100]$

Figure: Quadratic $\sigma_3$ vs. Failure Probability

$$h(x) = 0.85961511 + 6.15174179x^{0.13562741} e^{-0.26987804x} + 0.35691471xe^{-0.10525011x}$$
What is the relationship between the failure probability of
\[ f = c_1 + c_2 x^{99} + c_3 x^{100} \] and \[ \frac{\sigma^2}{\sigma_1^2} \], recalling that \( c_i \sim N(0, \sigma_i) \)?
Experimental Consideration

What is the relationship between the failure probability of 
\[ f = c_1 + c_2 x^{99} + c_3 x^{100} \] 
and \( \frac{\sigma_2}{\sigma_1} \), recalling that \( c_i \sim N(0, \sigma_i) \)?

Method:

- 100 values of \( \sigma_2 \) in \([0.1, 60]\)
- 1,000,000 trials per ratio
- generate \( c_1 \) and \( c_3 \) from standard Gaussian distributions, and \( c_2 \) from \( N(0, \sigma_2) \) each trial
Varying the standard deviation of $c_2$:

$$h(x) = -0.06450709 + 0.18826155x^{0.55247034}e^{-0.15034146x} - 1.03096168xe^{-1.09906311x}$$
Varying the standard deviation of $c_1$:

Figure: $\sigma_1$ vs. Failure Probability
New Experimental Questions

- Can we simplify the fit functions in some way?

Idea:
- Could using multiple simple piecewise functions approximate the failure probabilities?
- Can we extract meaning from the coefficients of the fit functions?
- Do the coefficients have a relationship to the exponent spread of the polynomial?
- Can we transform the fit functions into upper bounds?
- Can we find specific coefficients that upper bound the failure probabilities for all exponent spreads?
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**Figure:** Piecewise linear and $x^{-k}$ fit functions for failure probability vs. $\sigma$
Experimental Consideration

What is the minimum slope that upper bounds the failure probability when \( \sigma_2 \leq 1 \)?

\[
f(x) = c_1 + c_2 x + c_3 x^2
\]

Figure: Linear upper bound and fit lines for failure probability vs. \( \sigma \leq 1 \)
Experimental Consideration

What is the minimum slope that upper bounds the failure probability when $\sigma_2 \leq 1$, and what is its relationship to the trinomial’s spread?
Experimental Consideration

*What is the minimum slope that upper bounds the failure probability when \( \sigma_2 \leq 1 \), and what is its relationship to the trinomial’s spread?*

**Method:**

- 10 exponent ratios in \([0.1, 1]\)
  - 10 values of \( \sigma_2 \) in \([0.1, 1]\)
  - 100,000 trials per \( \sigma_2 \)
- generate \( c_1 \) and \( c_3 \) from standard Gaussian distributions, and \( c_2 \) from \( N(0, \sigma_2) \) each trial
- find upper bound curve of form \( g(x) = ax \)
- per trinomial exponent ratio, average 10 values of \( a \)
Figure: Minimum slopes for upper bound line vs. trinomial exponent ratio
Piecewise Variance Ratio: $\sigma_2 \leq 1$ Results

$$g(x) = a\sqrt{\max\left(\alpha_1, \alpha_2 - \alpha_1\right)}$$

Figure: Minimum slopes for upper bound line vs. trinomial exponent ratio
Experimental Consideration

Finding a function of the form \( g(x) = ax^{-k} \) which is an upper bound for failure probability when \( \sigma_2 \geq 1 \).
Piecewise Variance Ratio: $\sigma_2 \geq 1$

Experimental Consideration

Finding a function of the form $g(x) = ax^{-k}$ which is an upper bound for failure probability when $\sigma_2 \geq 1$.

**Method:**

- 10 exponent ratios in $[0.1, 1]$
  - 10 values of $\sigma_2$ in $[1, 20]$
  - 1,000,000 trials per $\sigma_2$
- generate $c_1$ and $c_3$ from standard Gaussian distributions, and $c_2$ from $N(0, \sigma_2)$ each trial
- fit data to $g(x) = ax^{-k}$ using scipy’s `curve_fit` function
- increment $k$ until $g$ is an upper bound curve
- per exponent ratio, average 10 values of $k$
Piecwise Variance Ratio: $\sigma_2 \geq 1$ Results I

Figure: Upper bound constants and exponents vs. trinomial exponent ratios
Experimental Consideration

What is the minimum upper bound curve of the form $g(x) = ax^{-0.9}$ for failure probability when $\sigma_2 \geq 1$. 
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**Method:**

- 10 exponent ratios in $[0.1, 1]$
- 10 values of $\sigma_2$ in $[1, 20]$
- 1,000,000 trials per $\sigma_2$
- generate $c_1$ and $c_3$ from standard Gaussian distributions, and $c_2$ from $N(0, \sigma_2)$ each trial
- fit data to $g(x) = ax^{-0.9}$ using scipy’s `curve_fit` function
- increment $a$ until $g$ is an upper bound curve
- select maximum $a$
Piecewise Variance Ratio: $\sigma_2 \geq 1$ Results II

$$g(x) = 6.5x^{-0.9}$$

Figure: $f(x) = c_1 + c_2x + c_3x^2$

Figure: $f(x) = c_1 + c_2x^{99} + c_3x^{100}$
Further Work

- Tighter bound lines (especially for $\sigma \geq 1$)?
- Coefficient meaning for $\sigma \geq 1$?
  - Possible dependence on spread?
- Can we establish theoretical bounds that support these experimental results?
- Can we otherwise characterize the polynomials which fail?
Thank you for listening!

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