Degenerate and Non-Degenerate Embedding Dimensions of Neural Codes

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Introduction

Motivation

Place Cells

Place Cells are a collection of neurons that relay information on an organism’s spacial position within a location.

Ultimate Goal:

Understand what types of spaces a given neural code can represent.
Background:

Neural Code

A **Neural Code** is a collection of code words that represent the possible combinations of neurons firing within a set of given receptive fields.

Set of Receptive Fields: $\mathcal{U} = \{U_1, U_2, U_3\}$

$\mathcal{C} = \{000, 100, 010, 001, 110, 011\}$

$A^\mathcal{U}_{011} = \{ (U_2 \cap U_3) \setminus U_1 \}$

$\mathcal{C} = \text{code}(\mathcal{U}, \mathbb{R}^2)$
Convexity

A set is **convex** if for any two points within the set, the line segment between them can be drawn wholly within the set.

**Figure:** Convex (left), Non-convex (right) $d=2$
Closed/Open Convexity in Neural Codes

A closed/open convex neural code is a neural code that can be represented as a set of convex receptive fields where the receptive fields are closed/open sets.

Figure: $U_1$: (Blue) ; $U_2$: (Red)

Example

$U = \{U_1, U_2\}$

$C_C = \{00, 10, 01, 11\}$

$C_C = code(cl(U), \mathbb{R}^1)$

$C_O = \{00, 10, 01\}$

$C_O = code(int(U), \mathbb{R}^1)$
Open/Closed — Embedding Dimension

The **open/closed embedding dimension** is the lowest dimension such that a neural code has an open/closed-convex realization.

**Example**

\[ C_\emptyset = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\} \]
Project:
Find and classify the relationships between the minimal open and closed embedding dimensions.

Figure: Closed \((\mathcal{U}, \mathbb{R}^2)\), Open \((\mathcal{U}, \mathbb{R}^2)\), Open & Closed \((\mathcal{U}, \mathbb{R}^3)\) for \(C = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\}\)
Non-degenerate Realizations

A realization \((\mathcal{U} = \{ U_i \}, \mathbb{R}^d)\) is **non-degenerate** if:

1. For any arbitrary open set \(S_o \subseteq \mathbb{R}^d\) where \(S_o \neq \emptyset\) and all \(A_{c_j}^U\) where \(A_{c_j}^U \cap S_o \neq \emptyset\), it is also the case that \(\text{int}(A_{c_j}^U \cap S_o) \neq \emptyset\)

2. For all non-empty \(\sigma \subseteq [n] = \{1, 2, \ldots, n\}\), \((\cap_{i \in \sigma} \partial U_i) \subseteq \partial (\cap_{i \in \sigma} U_i)\)

**Figure:** Left \((\mathcal{U}, \mathbb{R}^1)\); Right: \((\mathcal{U}, \mathbb{R}^2)\)
Classification of Embedding Dimensions

- Open Non-Degenerate Embedding Dimension
  \[ \text{dim}_{O_{nd}}(C) = d_{O_{nd}} \]

- Closed Non-Degenerate Embedding Dimension
  \[ \text{dim}_{C_{nd}}(C) = d_{C_{nd}} \]

- Open Degenerate Embedding Dimension
  \[ \text{dim}_{O_{d}}(C) = d_{O_{d}} \]

- Closed Degenerate Embedding Dimension
  \[ \text{dim}_{C_{d}}(C) = d_{C_{d}} \]

\[ \text{dim}_{C_{nd}}(C) \leftrightarrow \? \rightarrow \text{dim}_{O_{nd}}(C) \]

\[ \text{dim}_{C_{d}}(C) \leftrightarrow \? \rightarrow \text{dim}_{O_{d}}(C) \]
Question 1:
What is the relation between the open non-degenerate and the closed non-degenerate embedding dimension?

Non-degenerate Embedding Dimension
The non-degenerate embedding dimension of \( \mathcal{C} \) is the lowest dimension such that a convex non-degenerate realization can be made.

Lemma 1: (J. Cruz, C. Giusti, V. Itskov, and B. Kronholm)
If \( \mathcal{U} = \{U_i\} \) is a convex and non-degenerate cover, then:

- \( U_i \) are open \( \Rightarrow \) \( \text{code}(\mathcal{U}, \mathbb{R}^d) = \text{code}(\text{cl}(\mathcal{U}), \mathbb{R}^d) \);
- \( U_i \) are closed \( \Rightarrow \) \( \text{code}(\mathcal{U}, \mathbb{R}^d) = \text{code}(\text{int}(\mathcal{U}), \mathbb{R}^d) \).

This states all codes that have a non-degenerate convex realization are both open and closed convex.
Lemma 1: (J. Cruz, C. Giusti, V. Itskov, and B. Kronholm)

If $\mathcal{U} = \{U_i\}$ is a convex and non-degenerate cover, then:

- $U_i$ are open $\implies code(\mathcal{U}, \mathbb{R}^d) = code(cl(\mathcal{U}), \mathbb{R}^d)$;
- $U_i$ are closed $\implies code(\mathcal{U}, \mathbb{R}^d) = code(int(\mathcal{U}), \mathbb{R}^d)$.

Theorem 1: (Chan-Johnston)

Given a neural code $\mathcal{C}$, the non-degenerate closed embedding dimension $\dim_{C_{nd}}(\mathcal{C})$ and the non-degenerate open embedding dimension $\dim_{O_{nd}}(\mathcal{C})$ are equal to the same dimension $d$. 
Question 2:
What is the relation between the non-degenerate and the degenerate embedding dimension?

Degenerate Embedding Dimension

The **degenerate embedding dimension** of $C$ is the lowest dimension such that a convex realization can be made regardless of degeneracy.

\[ C = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\} \]

**Figure:** Closed $(U, \mathbb{R}^2)$, Open $(U, \mathbb{R}^2)$, Open & Closed $(U, \mathbb{R}^3)$
Theorem 2: (Chan-Johnston)

Given a neural code $C$, the non-degenerate embedding dimension $\dim_{nd}(C)$ is greater than or equal to the degenerate open and closed embedding dimension, $\dim_{Od}(C)$ and $\dim_{Cd}(C)$.

Importance

The non-degenerate embedding dimension acts as an upper bound for all other embedding dimensions.
Question 3:
What is the relation between open and closed degenerate embedding dimension?

Theorem 3: (Chan-Johnston)
If $\mathcal{U}$ is a convex and degenerate cover:

1. $(\mathcal{U}, \mathbb{R}^d)$ is an open realization of a neural code $\mathcal{C}$ $\implies$ $(\text{cl}(\mathcal{U}), \mathbb{R}^d)$ is not a closed realization of $\mathcal{C}$

2. $(\mathcal{U}, \mathbb{R}^d)$ is an closed realization of a neural code $\mathcal{C}$ $\implies$ $(\text{int}(\mathcal{U}), \mathbb{R}^d)$ is not a open realization of $\mathcal{C}$. 
Conjecture 1: (Chan-Johnston)

Let $\mathcal{C}$ have an embedding dimension of $d$. For all convex realizations with an embedding dimension greater than their respective neural code's embedding dimension, $(\mathcal{U}, \mathbb{R}^{d_\theta \geq d})$, is homotopy equivalent to a realization of the neural code in the code's embedding dimension where the intermediate realizations that have undergone a continuous deformation are valid realizations (convex and the code remains unchanged).
**Theorem 4: (Chan-Johnston)**

Let $\mathcal{C}$ be a neural code that satisfies Conjecture 1. Then, if a neural code $\mathcal{C}$ is open and closed convex and there exists a non-degenerate realization, then:

1. $\dim_{nd}(\mathcal{C}) = \dim_{O_d}(\mathcal{C})$,
2. $\dim_{O_d}(\mathcal{C}) \geq \dim_{C_d}(\mathcal{C})$.

**Figure:** Continuous deformation of a non-degenerate realization

$\mathcal{C} = \{110, 101, 011, 100, 010, 001\}$
**Theorem 4: (Chan-Johnston)**

Let $\mathcal{C}$ be a neural code that satisfies Conjecture 1. Then, if a neural code $\mathcal{C}$ is open and closed convex and there exists a non-degenerative realization, then:

1. $\dim_{nd}(\mathcal{C}) = \dim_{Od}(\mathcal{C})$, 
2. $\dim_{Od}(\mathcal{C}) \geq \dim_{Cd}(\mathcal{C})$.

**Figure:** Continuous deformation of a non-degenerative realization

$\mathcal{C} = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\}$
$\dim_{C_{nd}}(C) \underset{=} \geq \dim_{O_{nd}}(C)$

$\dim_{C_{d}}(C) \underset{\leq_*} \rightarrow \dim_{O_{d}}(C)$

**Figure:** *Assumes Conjecture 1 and the existence of a non-degenerate realization*

**Possible Future Questions:**

- Prove conjecture 1.
- If a neural code is open and closed convex, then does there exist a non-degenerate realization?
THANK YOU!

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