Bounds for Coefficients of the $f(q)$ Mock Theta Function and Applications to Partition Ranks (Part 2)

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We utilize our effective bound on \( \alpha(n) \) to resolve the following conjecture:

**Conjecture (Hou and Jagadeesan [2], 2017)**

If \( r = 0 \) (resp. \( r = 1 \)), then we have that

\[
N(r, 2; a)N(r, 2; b) > N(r, 2; a + b)
\]

for all \( a, b \geq 11 \) (resp 12).

Hou and Jagadeesan demonstrated a similar result for the modulo-three rank-counting functions \( N(r, 3; n) \) for \( r = 0, 1, 2 \), but their methods do not work modulo two.
Growth of $N(r, 2; n)$

**Theorem (Gomez-Zhu)**

For $n \geq 4$,

$$N(r, 2; n) = \frac{H(n)}{36l(n)^2} \left(1 - \frac{1}{l(n)}\right) + (-1)^r R_2(n)$$

where $H(n) := \pi^2 \sqrt{3} e^{l(n)}$ and

$$|R_2(n)| \leq (8.17 \times 10^{30}) e^{l(n)/2}$$
We will make use of an effective bound on the partition function due to Lehmer:

**Theorem (Lehmer, 1938)**

*For all* \( n \geq 1 \),

\[
p(n) = \frac{2\sqrt{3}}{24n-1} \left(1 - \frac{1}{l(n)}\right) e^{l(n)} + E_p(n)
\]

*where* \( |E_p(n)| \leq (1313)e^{l(n)/2} \).
We substitute the asymptotic formulas for \( p(n) \) and \( \alpha(n) \) into the relation

\[
N(r, 2; n) = \frac{p(n) + (-1)^r \alpha(n)}{2}
\]

and then bound the resulting error

\[
R_2(n) := (-1)^{n-1} \frac{\pi}{\sqrt{6} l(n)} e^{l(n)/2} + \frac{1}{2} (E_p(n) + E(n)).
\]
Bounding $N(r, 2; n)$

We will use the previous theorem to prove the following crucial inequalities:

Lemma (Gomez-Zhu)

For $r = 0$ (resp. $r = 1$), we have that

$$\frac{H(n)}{36l(n)^2} \left(1 - \frac{1}{l(n)}\right)^2 < N(r, 2; n) < \frac{H(n)}{36l(n)^2} \left(1 - \frac{1}{l(n)^2}\right)$$

for all $n \geq 16$ (resp. 15).

This lemma places $N(r, 2; n)$ into a “nice” window, one which we manipulate to resolve the conjecture.
By our previous theorem,

\[ \frac{H(n)}{36l(n)^2} \left( 1 - \frac{1}{l(n)} \right) - |R_2(n)| < N(r, 2; n) \]

and

\[ N(r, 2; n) < \frac{H(n)}{36l(n)^2} \left( 1 - \frac{1}{l(n)} \right) + |R_2(n)|. \]

Thus, we can bound \( N(r, 2; n) \) for large enough \( n \)

\[ \frac{H(n)}{36l(n)^2} \left( 1 - \frac{1}{l(n)} \right)^2 < N(r, 2; n) < \frac{H(n)}{36l(n)^2} \left( 1 - \frac{1}{l(n)^2} \right) \]

so long as the coefficient of \( e^{l(n)} \) bounding \( |R_2(n)| \) is not too large.
Sketch of Proof

How large is too large? Given that $|R_2(n)| \leq (8.17 \times 10^{30}) e^{l(n)}$, we need $n$ large enough to satisfy

$$8.17 \times 10^{30} < \frac{\pi^2 \sqrt{3}}{36 l(n)^3} \left(1 - \frac{1}{l(n)}\right) e^{l(n)/2}.$$ 

Computation shows that $n > 4647$ will do, but we require our bounds to hold for significantly smaller $n$ to resolve the conjecture.

We thus analyze the remaining $n < 4647$ using the Online Encyclopedia of Integer Sequences, which contains the values of $p(n)$ and $\alpha(n)$ for $1 \leq n \leq 10^4$, and find that $N(r, 2; n)$ falls into our window for $n \geq 15$ when $r = 0$ (resp. $n \geq 16$ when $r = 1$).
We now prove the complete conjecture. Assume $16 \leq a \leq b$ and let $b = Ca$ where $C \geq 1$. We have just demonstrated that

$$N(r, 2; a)N(r, 2; Ca) > \frac{H(a)H(Ca)}{1296l(a)^2l(Ca)^2} \left(1 - \frac{1}{l(a)}\right)^2 \left(1 - \frac{1}{l(Ca)}\right)^2$$

and

$$N(r, 2; a + Ca) < \frac{H(a + Ca)}{36l(a + Ca)^2} \left(1 - \frac{1}{l(a + Ca)^2}\right).$$

Thus, we need only find $a$ such that our lower bound for $N(r, 2; a)N(r, 2; Ca)$ exceeds our upper bound for $N(r, 2; a + Ca)$. 

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This is equivalent to finding $a$ such that

$$e^{T_a(C)} > \frac{12\sqrt{3}l(a)^2 l(Ca)^2}{\pi^2 l(a + Ca)^2} S_a(C),$$

where

$$T_a(C) := l(a) + l(Ca) - l(a + Ca)$$

and

$$S_a(C) := \frac{\left(1 - \frac{1}{l(a+Ca)^2}\right)}{\left(1 - \frac{1}{l(a)}\right)^2 \left(1 - \frac{1}{l(Ca)}\right)^2}.$$ 

Or, taking logarithms of both sides,

$$T_a(C) > \log \left(\frac{12\sqrt{3}l(a)^2 l(Ca)^2}{\pi^2 l(a + Ca)^2}\right) + \log S_a(C).$$
We first observe that, as functions of $C$, $T_a$ is strictly increasing and $S_a$ is strictly decreasing, so we need only find $a$ which satisfy our inequality for $C = 1$

$$T_a(1) > \log \left( \frac{12\sqrt{3}l(a)^2 l(Ca)^2}{\pi^2 l(a + Ca)^2} \right) + \log S_a(1).$$

We then make use of the fact that $l(Ca)^2/l(a + Ca)^2 \leq 1$ for all $a$ since $l(a + Ca) > l(Ca)$ to reduce our inequality to

$$T_a(1) > \log \left( \frac{12\sqrt{3}l(a)^2}{\pi^2} \right) + \log S_a(1).$$
For which $a$ is this final relation true? We calculate $T_a(1)$ and $S_a(1)$ and find that $a \geq 16$ suffice, and thus the conjecture is proven for such $a, b \geq 16$.

The remaining cases of $11 \leq a, b \leq 15$ (resp. $12 \leq a, b \leq 15$) for $r = 0$ (resp. $r = 1$) are then checked manually by comparing $N(r, 2; a)$, $N(r, 2; b)$, and $N(r, 2; a + b)$. 

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Further Speculation

With this result, we might ask if we can obtain similar convexity results for other moduli? That is, do we have, for $t > 3$ and $0 \leq r < t$,

$$N(r, t; a)N(r, t; b) > N(r, t; a + b)$$

for all $a, b \geq C(t)$, where $C(t) > 0$ is an explicit constant depending only on the modulus $t$?

If we were able to find finite algebraic formulas describing $N(r, t; n)$ analogous to ours for larger $t$, this conjecture would be resolved as in the case of $t = 2$. However, no such formulas are yet known.
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