Parallel SSEP and a Casimir Element of $\mathfrak{so}_{2n}$

Mark Landry, Andrew Park

Texas A&M University

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Goals

1. Review relevant concepts from Markov processes and Lie algebras.
2. Show the process used to start from a Casimir element of $\mathfrak{so}_{2n}$ and arrive at a generator matrix.
3. Describe expected properties of the Markov process given the generator matrix.
A **Markov process** is a continuous-time physical process with a discrete number of states where states jump to other states at random times and the probabilities depend only on the present state, not past states.

A **generator matrix** encodes the jump rates between states. It has the properties:

- Each row sums to 0.
- All diagonal entries are non-positive.
- All off-diagonal entries are non-negative.

**Proposition**

Let $Q$ be a generator matrix, and let $q_{xy}$ be the $(x, y)$ entry of $Q$. Let $T_x$ be the holding time at state $x$. If $q_{xx} \neq 0$, then

$$P(X_{T_x} = y | X_0 = x) = \frac{q_{xy}}{-q_{xx}}.$$
A Simple Generator Matrix Example

\[
\begin{pmatrix}
-2 & 1 & 1 \\
0 & 0 & 0 \\
1 & 0 & -1 \\
\end{pmatrix}
\]

Figure: Sample Markov Process
Example: Symmetric Simple Exclusion Process (SSEP)

- Introduced by Frank Spitzer (1970)
- 2-site generator matrix derived from a Casimir element of $\mathfrak{sl}_2$

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

- Can be expanded to $N$ sites

Figure: SSEP Configurations
\[ \mathfrak{so}_{2n}(\mathbb{C}) \text{ is the Lie algebra of matrices of the form:} \]

\[
\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \bigg| A, B, C, D \in \mathbb{C}^{n \times n}, A = -D^T, B^T = -B, C^T = -C \right\}
\]

Let \( E_{i,j} \) be the \( 2n \times 2n \) matrix with a 1 in the \((i, j)\) entry and 0 elsewhere.

- \( H_i = E_{i,i} - E_{n+i,n+i} \)
- \( X_{ij} = E_{i,j} - E_{n+j,n+i} \)
- \( Y_{ij} = E_{i,n+j} - E_{j,n+i} \)
- \( Z_{ij} = E_{n+i,j} - E_{n+j,i} \)

A Cartan-Weyl Basis of \( \mathfrak{so}_{2n} \) consists of \( H_i \) for all \( i \leq n \) and \( X_{ij}, X_{ji}, Y_{ij}, Z_{ij} \) for all \( i < j \leq n \).
Procedure to Obtain $\rho(\Omega)$

- **Dual Basis (Fulfills condition with respect to Killing form)**
  - Killing form $B(F, S) = (2n - 2) \text{Tr}(FS)$ is 1 if $S$ is dual basis counterpart of $F$, 0 for $S$ is in dual basis but not counterpart
  - $H_i \rightarrow \frac{1}{4n-4} H_i$
  - $X_{ij} \rightarrow \frac{1}{4n-4} X_{ji}$; $X_{ji} \rightarrow \frac{1}{4n-4} X_{ij}$
  - $Y_{ij} \rightarrow -\frac{1}{4n-4} Z_{ij}$; $Z_{ij} \rightarrow -\frac{1}{4n-4} Y_{ij}$

- **Casimir Element** $\Omega = \sum_i A_i A^i$, where $A_i$ is from basis and $A^i$ is counterpart in dual basis.

- **The $\rho$ Representation**
  - For $A \in \mathfrak{so}_{2n}$, $\rho_{\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}}(A) = \rho_{\mathbb{C}^{2n}}(A) \otimes \text{Id}_{2n} + \text{Id}_{2n} \otimes \rho_{\mathbb{C}^{2n}}(A)$
  - Example: $\rho_{\mathbb{C}^{2n}}(X_{12})$ is a $2n \times 2n$ matrix, $\rho_{\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}}(X_{12})$ is a $4n^2 \times 4n^2$ matrix.

- **Compute** $\rho(\Omega) = \sum_i \rho_{\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}}(A_i) \rho_{\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}}(A^i)$
The representation of the Casimir element \( \rho(\Omega) \), a \( 4n^2 \times 4n^2 \) matrix, can be written as a \( 2n \times 2n \) matrix of blocks each of size \( 2n \times 2n \) as:

\[
\rho(\Omega) = \frac{1}{2n-2} \begin{pmatrix}
D_1 & X_{21} & X_{31} & \cdots & X_{n,1} \\
X_{12} & D_2 & X_{32} & \cdots & X_{n,2} \\
X_{13} & X_{23} & D_3 & \cdots & X_{n,3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X_{1,n} & X_{2,n} & X_{3,n} & \cdots & D_n \\
0 & -Y_{12} & -Y_{13} & \cdots & -Y_{1,n} \\
Y_{12} & 0 & -Y_{23} & \cdots & -Y_{2,n} \\
Y_{13} & Y_{23} & 0 & \cdots & -Y_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Y_{1,n} & Y_{2,n} & Y_{3,n} & \cdots & 0 \\
\end{pmatrix}
\]

where \( D_i = (2n - 1)I + H_i \) and \( D_{n+i} = (2n - 1)I - H_i \) for \( i \leq n \).
From $\rho(\Omega)$ to $G_n$

$\rho(\Omega)$ is not yet a generator matrix, as its rows do not yet sum to 0. We need to use a correction term $C$ such that $\rho(\Omega) - C$ is a generator matrix.

**Procedure**

- Let $C$ be the diagonal matrix with entries $c_i$ such that $c_i = \sum_j \rho(\Omega)_{ij}$.
- Negate rows of $\rho(\Omega) - C$ to force all diagonal elements to be non-positive. Call this matrix $G_n$.

Note: Past research uses $C = kI$ for some $k \in \mathbb{C}$, and then conjugates with a diagonal matrix to arrive at a generator matrix. In the case of $so_{2n}$, this technique fails to arrive at a generator matrix with finite entries if $n > 2$. 
Lemma (Landry-Park)
The matrix $G_n$ is the generator of a Markov process.

Lemma (Landry-Park)
All non-zero off-diagonal entries of $G_n$ are equal.

First Properties of Markov Process:
- $G_n$ is $4n^2 \times 4n^2$, which implies a Markov process with $4n^2$ states.
- From past research on Lie algebras, Casimir representations describe a particle system with two sites.
Absorbing States

An **absorbing state** of a Markov process is a state that "absorbs," i.e. if the process lands there, it will never jump to another state. This is represented in a generator matrix by a row of 0’s.

**Lemma (Landry-Park)**
The Markov process with generator $G_n$ has $2n$ absorbing states.

![Figure: 2 Absorbing States in SSEP](image-url)
Definition (Landry-Park)

A **maximal choice row** is a row in which no other rows have a greater number of nonzero off-diagonal elements. The set of all maximal choice rows is called the **maximal choice set**.

Lemma (Landry-Park)

The generator $G_n$ has $2n$ maximal choice rows, each of which have $2n - 2$ non-zero off-diagonal entries.

**Figure: 4 Maximum Choice States**

**Figure: 6 Maximum Choice States**
Definition (Landry-Park)

A \textbf{pairwise row} is a row with exactly one nonzero off-diagonal entry. If pairwise row \( r \) has its nonzero off-diagonal entry at column \( s \), and \( s \) is a pairwise row with nonzero off-diagonal entry at column \( r \), rows \( r \) and \( s \) are called \textbf{pairwise states}.

Lemma (Landry-Park)

Any row in \( G_n \) is either an absorbing state, or a maximal choice row, or a pairwise state.

\[ \begin{align*}
\text{Figure: 2 Pairwise States from SSEP}
\end{align*} \]
Summary

Based on the properties we observe in $G_n$, a generator matrix derived from a Casimir element in $\mathfrak{so}_{2n}$, we expect the following properties in the corresponding Markov process:

- $4n^2$ states total.
- $2n$ absorbing states.
- $2n$ maximum choice states that can each jump to $2n - 2$ other maximum choice states.
- The remaining states split up into pairs that jump back and forth.
A Parallel SSEP with \( N \) sites is a system that has two separate 1-dimensional SSEPs with \( N \geq 2 \) sites each. Each site can either be empty or have a particle on it, and while particles can interact with neighboring sites on the same lattice, they cannot jump to the other lattice. A Parallel SSEP with 2 sites will be referred to as a Basic Parallel SSEP.

Type-1 Parallel SSEP is the simplest case of a Parallel SSEP; it is a Parallel SSEP with only 1 type of particle.

Figure: A possible state of a Type-1 Basic Parallel SSEP
Consider a system similar to the Type-1 Basic Parallel SSEP where we introduce a second type of particle, $1/2$ the mass of the first particle, to the upper lattice. This is the **Type-2 Basic Parallel SSEP** and has the following properties:

1. Heavier particles can not move as long as a lighter particle in the system can move.
2. If the two lattices are equal by mass, then the upper lattice allows fusion/fission which provides energy, possibly granting the lower lattice a free concurrent move.
3. Lighter particles have the “upper-class” property seen in other SSEP variants.
Type-2 Basic Parallel SSEP

Figure: The black particle is unable to move

Figure: The two lattices are balanced; the top particle can undergo fission

Figure: The red particle will switch places with the black particle
Consider a Basic Parallel SSEP where the lower lattice allows particles of mass 1, and the upper lattice allows particles of mass $\omega \in \{ \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1 \}$. We define the following properties:

1. **Mass Order Property**: A particle can only move if no lighter particles can move.

2. **Balance Property**: A set of balanced states exists in which the two lattices each have mass 1 and particles are able to undergo fusion and fission, defined respectively as donating mass to or taking mass from a neighboring site. These mass-preserving processes allow a concurrent move in the lower lattice.

3. **Class Property**: A particle in a non-balanced state is able to switch places with neighboring particles of higher mass.
**Type-m Basic Parallel SSEP**

**Definition (Landry-Park)**

A **Type-m Basic Parallel SSEP** is a Basic Parallel SSEP where the lower lattice allows particles of mass $1$, the upper lattice allows particles of mass $\omega \in \{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\}$, and the following properties hold: Mass Order Property, Balance Property, and Class Property.

![Diagram of a possible state of a Type-3 Basic Parallel SSEP with mass values: $\frac{1}{3}, \frac{2}{3}, 1$.]

**Figure:** A possible state of a Type-3 Basic Parallel SSEP with mass values: $\frac{1}{3}, \frac{2}{3}, 1$
We return to the generator $G_n$ of a Markov process, and its connection to our newly defined particle system:

**Theorem (Landry-Park)**

Let $m = n - 1$. The generator matrix of the Type-$m$ Basic Parallel SSEP is exactly $G_n$. 
**Lemma (Landry-Park)**

A Type-$m$ Basic Parallel SSEP has $4(m + 1)^2$ states made up of:

- $2(m + 1)$ absorbing states
- $2(m + 1)$ maximum-choice states
- $4(m + 1)^2 - 4(m + 1)$ pairwise states

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**Figure:** Different states of Type-3 Basic Parallel SSEP with mass values: $\frac{1}{3}, \frac{2}{3}, 1$
The following formula takes $L$, the generator matrix for the SSEP with 2 sites, and expands it to $L_N$, the generator matrix for the SSEP with $N$ sites:

$$L_N = \sum_{j=0}^{N-2} I \otimes \cdots \otimes I \otimes L \otimes I \otimes \cdots \otimes I$$

This formula also holds for the generator of a Parallel SSEP! Note that each lattice is expanded to $N$ sites, so $L_N$ gives $2N$ sites in total for a Parallel SSEP.
Subsystems

**Definition (Landry-Park)**

A **subsystem** of a Parallel SSEP with $N$ sites is a subset of the system such that it is a Basic Parallel SSEP. A Parallel SSEP with $N$ sites is made up of $N - 1$ overlapping subsystems which define the local properties of the entire system.

![Diagram of subsystems](image)

(a) Subsystem 1  
(b) Subsystem 2  
(c) Entire System

**Figure:** How subsystems make up the entire system
**Definition (Landry-Park)**

A **Type-m Parallel SSEP with N sites** is a Parallel SSEP with N sites such that every subsystem is a Type-m Basic Parallel SSEP.

![Diagram of a Type-3 Parallel SSEP with 5 sites]

**Figure**: A possible state of a Type-3 Parallel SSEP with 5 sites

**Remark**

*The total number of choices available to a state of a Type-m Parallel SSEP with N sites is just the sum of the number of choices of its subsystems.*
The Final Result

Theorem (Landry-Park)

Let $L = G_n$, the generator of a Type-$m$ Basic Parallel SSEP, and let:

$$L_N = \sum_{i=1}^{N-1} L \otimes \cdots \otimes L \otimes I_{i-1} \otimes I_1 \otimes \cdots \otimes I_{N-1-i},$$

$L_N$ is the generator of a Type-$m$ Parallel SSEP with $N$ sites.

Remark

*The $i^{th}$ term of the summation corresponds directly to the $i^{th}$ subsystem!*
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Jeffrey Kuan.
Stochastic duality of ASEP with two particle types via symmetry of quantum groups of rank two.

Frank Spitzer.
Interaction of markov processes.