Characterizing Codes with Three Maximal Codewords

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Overview

1. Biological Motivation
2. Definitions
3. Goal
4. Results
Biological Motivation

- Encode spatial structure
- Associate neurons to regions of space
- Precisely fire in receptive fields

Figure: Neuron firing pattern
Figure: Place Cell Example
Definitions

Neural Code

- A neural code $C$ on $n$ neurons is a set of subsets of $[n]$ (called codewords), i.e. $C \subseteq 2^n$.
- A maximal codeword in $C$ is a codeword that is not properly contained in any other codeword in $C$.
- Convex if it can be realized by a set of convex sets $U_1, U_2, \ldots, U_n \subseteq \mathbb{R}^d$. A code’s minimal embedding dimension is the smallest value of $d$ for which this is possible.

Example

$C = \{0, 1, 2, 3, 12, 23, 34, 13, 123, 234\}$, where $n = 4$. 

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**Simplicial Complexes**

An abstract *simplicial complex* on \( n \) vertices is a nonempty set of subsets (faces) of \([n]\) that is closed under taking subsets.

For a code \( C \) on \( n \) neurons, \( \Delta(C) \) is the smallest simplicial complex on \([n]\) that contains \( C \):

\[
\Delta(C) : = \{ \omega \subseteq [n] \mid \omega \subseteq \sigma \text{ for some } \sigma \in C \}.
\]

**Example**

\[
C = \{\emptyset, 1, 2, 3, 12, 23, 34, 13, 123, 234\}
\]

\[
\Delta(C) = \{123, 234, 13, 34, 23, 12, 24, 4, 3, 2, 1, \emptyset\}
\]
For a face $\sigma \in \Delta$, the link of $\sigma$ in $\Delta$ is the simplicial complex

$$Lk_\Delta(\sigma): = \{ \omega \subseteq \Delta \mid \sigma \cap \omega = \emptyset, \sigma \cup \omega \in \Delta \}.$$
Definitions Continued

**Contractible**

A set is *contractible* if it can be reduced to one of its points by a continuous deformation.

**Local Obstruction**

If $Lk_\Delta(\sigma)$ is NOT contractible and $\sigma \notin C$, a local obstruction occurs.

- $\sigma$ is an intersection of maximal codewords.
- Local obstructions imply non-convexity.
Max-intersection-complete

A code is *max-intersection-complete* if any arbitrary intersection of maximal codewords is in the original code.

- Max-intersection-complete $\Rightarrow$ convexity

Example

Max-intersection-complete code:

- $C = \{123, 234, 145, 23, 4, 1\}$

Non max-intersection-complete code:

- $C = \{123, 234, 145, 23\}$
Overarching Goal: Completely characterize codes with 3 maximal codewords

1. How to determine contractibility of triplewise intersections
2. Can we produce convex (open/closed) realizations for all codes
3. What are the embedding dimensions for the minimal/full codes
Lemma 4.7, (Curto et al.)

Let $\Delta$ be a simplicial complex. If $\sigma = \tau_1 \cap \tau_2$, where $\tau_1, \tau_2$ are distinct facets of $\Delta$, and $\sigma$ is not contained in any other facet of $\Delta$, then the $Lk_\sigma(\Delta)$ is not contractible.

Thus, we only have to look at the triplewise intersection.

**Case 1 - Link of Triplewise is Non-Contractible**
- All other cases

**Case 2 - Link of Triplewise is Contractible**
- Triplewise intersection is non-empty and there are exactly 2 distinct pairwise intersections
Contractible

$\Delta(C) = \{123, 124, 1356\}$ $F_1, F_2, F_3$

$F_1 \cap F_2 \cap F_3 = \{1\}$
$F_1 \cap F_2 = \{12\}$
$F_1 \cap F_3 = \{13\}$
$F_2 \cap F_3 = \{1\}$
Question: Does the absence of local obstructions imply convexity for codes with 3 maximal codewords?

Known Results

- max-intersection-complete $\Rightarrow$ convex $\Rightarrow$ no local obstructions
- max-intersection-complete $\not\Rightarrow$ convex ??? no local obstructions
  - $\not\Rightarrow$ for codes with 4 or more maximal codewords
Assume $C$ has no local obstructions

- **Case 1: Non-contractible link**
  - All intersections must be contained in $C$, thus max-intersection-complete

- **Case 2: Contractible link**
  - $C$ is not required to be max-intersection-complete in order to have no local obstructions. Thus, we must provide a convex realization that such codes are indeed convex.

**Recall:** contractible link if triplewise is nonempty & exactly 2 distinct pairwise
A minimal code is the smallest code with no local obstructions.

Example: \( C_{\text{min}}(\Delta) = \{123, 124, 1356, 13, 12, 1\} \)

Convex Realization for Case 2 Codes

Given a neural code \( C \) with three maximal codewords \( F_a, F_b, F_c \) such that \( F_a \cap F_b \cap F_c = \sigma \neq \emptyset \), \( F_a \cap F_b \neq \sigma \), \( F_b \cap F_c \neq \sigma \) and \( F_a \cap F_c = \sigma \). A convex (open/closed) realization of \( C_{\text{min}}(\Delta) \) can be constructed in dimension 1 such that the codewords appear in the following order:
**Question:** Do no local obstructions imply convexity for codes with 3 maximal codewords?

**Response:** Yes. Assume $\mathcal{C}$ has no local obstructions.

1. Case 1 - Contractible: Convex Realization
2. Case 2 - Non-contractible: Max-$\cap$-complete
Convex Realizations

Figure: Realization of $C_{\text{min}}(\Delta)$ in Dimension 2
Figure: Realization of the code $\mathcal{C} = \{F_a, F_b, F_c, F_a \cap F_b, F_b \cap F_c, c_1, c_2\}$

$C_{\text{min}}(\Delta) \subseteq \mathcal{C} \subseteq \Delta$
Embedding Dimension (Cruz et al.)

- For a minimal code, $C_{\text{min}}(\Delta)$, consisting of only max codewords and their intersections, there exists an open/closed convex realization of $C_{\text{min}}(\Delta)$ in $\mathbb{R}^{k-1}$, where $k$ is the number of max codewords.
- Furthermore, by going to $\mathbb{R}^k$, you can get a realization of any code of the same simplicial complex that contains the minimal code.

Example

- $C_{\text{min}}(\Delta) = \{123, 124, 1356, 13, 12, 1\}$ (Realizable in 2D)
- For a code, $C$, such that $C_{\text{min}}(\Delta) \subseteq C \subseteq \Delta$ (Realizable in 3D)
  - $C = \{123, 124, 1356, 13, 12, 1, 2, 3, 4\}$
Embedding Dimension

- Expansion upon the result from Cruz et al:

**Table:** Minimal embedding dimension of $C_{min}(\Delta)$ based on the number of pairwise intersections distinct from the triplewise

<table>
<thead>
<tr>
<th>Embedding Dimension</th>
<th>Pairwise Intersections</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Theorem 3.6 (Johnston - Spinner)**

If $C$ is a neural code with exactly 3 maximal codewords, then the minimal embedding dimension is at most 2.
Thank you for listening!

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