THE DISTRIBUTION OF SHORT ORBITS OF SINGULAR MODULI

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Abstract. We study the asymptotic distribution of weak Maass forms averaged over short orbits of Heegner points. Under a mild condition on the growth of the size of these orbits, we give an asymptotic formula with a power-saving error term for these averages. We apply our results to compute the limiting distribution of short orbits of singular moduli.

1. Introduction and statement of results

1.1. Overview. In this paper we study the asymptotic distribution of weak Maass forms averaged over short orbits of Heegner points.

To summarize our results, let $k \geq 0$ be an integer and $M_{-2k}^!(1)$ be the space of weakly holomorphic modular forms of weight $-2k$ and level one. Define the differential operator $\mathcal{D}^k$ by

$$\mathcal{D}^k f := \frac{1}{(4\pi)^k} R_{-2} R_{-4} \cdots R_{-2k} f, \quad k \geq 1,$$

and $\mathcal{D}^0 f = f$, where $R_t$ is the Maass weight raising operator

$$R_t f := 2i \frac{\partial}{\partial z} + \frac{t}{y}, \quad t \in \mathbb{Z}.$$

The operator $\mathcal{D}^k$ maps $M_{-2k}^!(1)$ to the space of weight zero weak Maass forms of level one.

Let $d < -4$ be an odd fundamental discriminant and $\Lambda_d$ be the set of Heegner points of discriminant $d$ on the modular curve $X_0(1)$. The class group $G_d$ acts simply transitively on $\Lambda_d$.

For each $d$, choose a subgroup $H_d < G_d$ and a Heegner point $\tau_{0,d} \in \Lambda_d$. Consider the $H_d$-orbit

$$H_d \cdot \tau_{0,d} = \{ \tau^\sigma_{0,d} : \sigma \in H_d \}$$

and the corresponding average

$$\text{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d}) := \frac{1}{|H_d|} \sum_{\sigma \in H_d} \mathcal{D}^k f(\tau^\sigma_{0,d}).$$

We will give an asymptotic formula with a power-saving error term for $\text{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d})$ as $|d| \to \infty$ for sequences of subgroups $(H_d)$ satisfying a mild growth condition; see Theorem 1.1 and Corollary 1.3. We then apply this result when $k = 0$ and $f = j$ is the modular $j$-function to compute the limiting distribution of averages of short orbits of singular moduli; see Corollary 1.4.
1.2. Quadratic forms and Heegner points. We fix the following setup concerning quadratic forms and Heegner points.

Let \( d \leq -4 \) be an odd fundamental discriminant and \( \mathcal{Q}_d \) be the set of positive definite, primitive, integral binary quadratic forms

\[
Q(X, Y) = [a_Q, b_Q, c_Q](X, Y) = a_QX^2 + b_QXY + c_QY^2
\]

of discriminant \( b_Q^2 - 4a_Qc_Q = d \). There is a (right) action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathcal{Q}_d \) defined by

\[
Q = [a_Q, b_Q, c_Q] \mapsto Q\gamma = [a_{Q\gamma}, b_{Q\gamma}, c_{Q\gamma}]
\]

where

\[
\begin{align*}
a_{Q\gamma} &= a_Q\alpha^2 + b_Q\alpha\gamma + c_Q\gamma^2, \\
b_{Q\gamma} &= 2a_Q\alpha\beta + b_Q(\alpha\delta + \beta\gamma) + 2c_Q\gamma\delta, \\
c_{Q\gamma} &= a_Q\beta^2 + b_Q\beta\delta + c_Q\delta^2.
\end{align*}
\]

The set \( \mathcal{Q}_d / \text{SL}_2(\mathbb{Z}) \) is a finite abelian group with respect to Gauss’s law of composition of forms. Let \( \mathbb{G}_d = \mathcal{Q}_d / \text{SL}_2(\mathbb{Z}) \) be the class group and \( h(d) = |\mathbb{G}_d| \) be the class number.

To each form \( Q \in \mathcal{Q}_d \) we associate a Heegner point \( \tau_Q \) which is the root of \( Q(X, 1) \) given by

\[
\tau_Q = \frac{-b_Q + \sqrt{D}}{2a_Q} \in \mathbb{H}.
\]

We write \( x_Q := \text{Re}(\tau_Q) \) and \( y_Q := \text{Im}(\tau_Q) \). The Heegner points \( \tau_Q \) are compatible with the action of \( \text{SL}_2(\mathbb{Z}) \) in the sense that if \( \gamma \in \text{SL}_2(\mathbb{Z}) \), then

\[
\gamma(\tau_Q) = \tau_{Q\gamma^{-1}}.
\]

We define the set of Heegner points of discriminant \( d \) by

\[
\Lambda_d := \{ \tau_Q : [Q] \in \mathcal{Q}_d / \text{SL}_2(\mathbb{Z}) \}.
\]

Given two forms \( Q, Q' \in \mathcal{Q}_d \), let \( Q \circ Q' \) denote their composition. The group \( \mathbb{G}_d \) acts simply transitively on \( \Lambda_d \) by

\[
[Q'] \cdot \tau_Q = \tau_{Q \circ Q'}.
\]

We will also denote this action by \( \tau_Q^{[Q']} \).

Recall that a form \( Q \in \mathcal{Q}_d \) is reduced if

\[
|b_Q| \leq a_Q \leq c_Q,
\]

and if, in addition, \( |b_Q| = a_Q \) or \( a_Q = c_Q \), then \( b_Q \geq 0 \). Each class in \( \mathcal{Q}_d / \text{SL}_2(\mathbb{Z}) \) contains a unique reduced form. Let \( \mathcal{Q}_d^{\text{red}} \) denote the set of reduced forms representing the classes in \( \mathcal{Q}_d / \text{SL}_2(\mathbb{Z}) \). If \( Q \in \mathcal{Q}_d^{\text{red}} \), then the corresponding Heegner point \( \tau_Q \) lies in the standard fundamental domain \( \mathcal{F} \) for \( \text{SL}_2(\mathbb{Z}) \).

Finally, let \( \widehat{\mathbb{G}}_d \) be the group of characters \( \chi : \mathbb{G}_d \to S^1 \) and \( H_d^+ < \widehat{\mathbb{G}}_d \) be the subgroup of characters \( \chi \) which restrict to the identity on a subgroup \( H_d < \mathbb{G}_d \).
1.3. **Bounds for $L$–functions.** Let $\chi$ be a character of $G_d$ and $\chi_d$ be the quadratic Dirichlet character of conductor $d$. Let $g$ be an arithmetically normalized Hecke-Maass form for $SL_2(\mathbb{Z})$ with eigenvalue $\lambda_g = 1/4 + t_0^2$ and $\Theta_{\chi}$ be the theta function of weight one and level $|d|$ associated to $\chi$. Let $L(g \otimes \chi, s)$ be the Rankin-Selberg $L$-function of $g \otimes \theta_{\chi}$, $L(\chi, s)$ be the $L$–function of $\chi$, $L(\chi_d, s)$ be the $L$–function of $\chi_d$ and $\zeta(s)$ be the Riemann zeta function. We assume bounds of the form

$$L(g \otimes \chi, 1/2) \ll_{\epsilon} \lambda_g^{B_1+\epsilon}|d|^{\delta_1+\epsilon},$$

$$L(\chi, 1/2 + it) \ll_{\epsilon} (1/4 + t^2)^{B_2+\epsilon}|d|^{\delta_2+\epsilon},$$

$$L(\chi_d, 1/2 + it) \ll_{\epsilon} (1/4 + t^2)^{B_3+\epsilon}|d|^{\delta_3+\epsilon},$$

$$\zeta(1/2 + it) \ll_{\epsilon} (1/4 + t^2)^{B_4+\epsilon}$$

for some absolute constants $B_1, B_2, B_3, B_4 > 0$, $0 < \delta_1 < 1/2$ and $0 < \delta_2, \delta_3 < 1/4$.

By Harcos and Michel [6], the bound (2) holds for some sufficiently large $B_1 > 0$ and $\delta_1 = 1499/3000$. By Duke, Friedlander and Iwaniec [3], the bound (3) holds with $B_2 = 5$ and $\delta_2 = 1/4 - 1/23041$. By Young [13], the bound (4) holds with $B_3 = 1/12$ and $\delta_3 = 1/6$. By Bourgain [1], the bound (5) holds with $B_4 = 13/168$. We note that stronger bounds in either the spectral or conductor aspect may exist; we stated here bounds in which both $B_i$ and $\delta_i$ are given explicitly, with the exception of (2), in which case an explicit $B_1$ has not yet been given (see Remark 1.2).

The Lindelöf Hypothesis implies that the bounds (2) – (5) hold with $B_1 = B_2 = B_3 = B_4 = \delta_1 = \delta_2 = \delta_3 = 0$.

1.4. **Main results.** The following is our main result.

**Theorem 1.1.** Let $k \geq 0$ be an integer and $f \in M_{-2k}(1)$ be a weakly holomorphic modular form with Fourier expansion

$$f(z) = \sum_{m=0}^{\infty} a(-m)q^{-m} + \sum_{m=1}^{\infty} a(m)q^m, \quad q := e(z) = e^{2\pi iz}.$$

For each $d$, choose a subgroup $H_d < G_d$ and a Heegner point $\tau_{0,d} = \tau_{[Q_{\tau_{0,d}}]} \in \Lambda_d$. There is an absolute constant $0 < \delta < 1/2$ given by (6) such that

$$Av_{H_d}(D^k f, \tau_{0,d}) = \frac{1}{h(d)} \sum_{Q \in Q_{\text{red}}^{\text{rel}}} C(\tau_{0,d}, Q) \sum_{m=0}^{N_{\infty}} a(-m)c_k(m; y_Q) e(-m\tau_Q)$$

$$+ \frac{3}{\pi} \beta_k(f) + O_{\epsilon}(|H_d|^{-1}|d|^{\delta+\epsilon})$$

as $|d| \to \infty$ where

$$C(\tau_{0,d}, Q) := \sum_{\chi \in \hat{H}_d} \overline{\chi}(Q_{\tau_{0,d}}^{-1} \circ Q),$$

$$c_k(m, y) := \sum_{j=0}^{k} (-1)^j(k + j)!m^{k-j}$$

$$(4\pi y)^{j(k-j)!},$$

$$(128) = 5, (163) = 2.$$
and
\[ \beta_k(f) := \int_{\text{reg}} \mathcal{D}^k f(z) d\mu \]
is the regularized integral defined by (22). Assuming the Lindelöf Hypothesis, we have \( \delta = 9/20 \).

**Remark 1.2.** Let \( B_1, B_2, B_3, B_4 \) and \( \delta_1, \delta_2, \delta_3 \) be as in the bounds (2) – (5). Then the constant \( \delta = \delta(B_1, B_2, B_3, B_4, \delta_1, \delta_2, \delta_3, \varepsilon) \) in Theorem 1.1 is given by
\[
\delta = \begin{cases} 
\frac{1}{2} - \frac{1 - 2\delta_1}{4(2A + 1)}, & \delta_1 \geq 2\delta_2 \text{ and } A'(1 - 2\delta_1) \leq (1 - 4\delta_3)(2A + 1) \\
\frac{1}{2} - \frac{1 - 4\delta_2}{4(2A + 1)}, & \delta_1 \leq 2\delta_2 \text{ and } A'(1 - 4\delta_2) \leq (1 - 4\delta_3)(2A + 1) \\
\frac{1}{2} - \frac{1 - 4\delta_3}{4A'}, & A'(1 - 2\delta_1) \geq (1 - 4\delta_3)(2A + 1) \text{ and } A'(1 - 4\delta_2) \geq (1 - 4\delta_3)(2A + 1)
\end{cases}
\]
where
\[
A := [\max \{B_1/2 + 1 + \varepsilon, B_2 + 1/4 + 2\varepsilon\}] + 1, \\
A' := [B_3 + B_4 + 3/2 + 3\varepsilon] + 1.
\]

In order to give a numerical value for \( \delta \), we need numerical values for the constants \( B_i, \delta_i \). In Section 1.3 we listed values for all of these constants except \( B_1 \). The work of Harcos and Michel [6] gives a polynomial dependence on the spectral parameter in the bound (2), which ensures the existence of some sufficiently large \( B_1 \). However, it seems difficult to produce a numerical value for \( B_1 \).

If we impose a mild growth condition on the sequence of subgroups \( (H_d) \) then we can ensure a power-saving exponent in the error term of Theorem 1.1. In particular, this allows us to compute the limiting distribution of \( \text{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d}) \) as \( |d| \to \infty \).

**Corollary 1.3.** Let \( (H_d) \) be a sequence of subgroups such that \( |H_d| \gg |d|^{\eta} \) for some \( \eta > \delta \). Then
\[
\text{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d}) - \frac{1}{h(d)} \sum_{Q \in \mathcal{Q}_d^{\text{red}}, y_Q > \frac{\varepsilon}{\sqrt{\pi}} + |d|^{-(1/2 - \delta)}} C(\tau_{0,d}, Q) \sum_{m=0}^{N_{\infty}} a(-m)c_k(m, y_Q) e(-m\tau_Q) \\
= \frac{3}{\pi} \beta_k(f) + O_\varepsilon(|d|^{-(\eta - \delta) + \varepsilon})
\]
as \( |d| \to \infty \).

**1.5. Short orbits of singular moduli.** Let
\[
j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots
\]
be the classical modular \( j \)-function. Given a Heegner point \( \tau_{0,d} \in \Lambda_d \), the values
\[
S_d := \{ j(\tau_{0,d}^\sigma) : \sigma \in G_d \}
\]
are algebraic numbers called *singular moduli*. These numbers are $j$-invariants of CM elliptic curves and generate the Hilbert class field of $K_d = \mathbb{Q}(\sqrt{d})$.

The class group $G_d$ acts on the set of singular moduli $S_d$, and this action is equivariant in the sense that $j(\tau_{0,d})^\sigma = j(\tau_{0,d}^\sigma)$ for $\sigma \in G_d$. In particular,

$$\text{Av}_{H_d}(j, \tau_{0,d}) = \frac{1}{|H_d|} \sum_{\sigma \in H_d} j(\tau_{0,d})^\sigma$$

is the average of the $H_d$-orbit of the singular modulus $j(\tau_{0,d})$.

In [4], Duke determined the limiting distribution of traces of singular moduli, proving that

$$\frac{1}{h(d)} \left( \sum_{Q \in \mathbb{Q}_d^\text{red}} j(\tau_Q) - \sum_{Q \in \mathbb{Q}_d^\text{red}, y_Q > 1} e(-\tau_Q) \right) \to 720$$

as $|d| \to \infty$. In particular, this resolved a conjecture of Bruinier, Jenkins and Ono [2] regarding the convergence of a Rademacher-type series expression for traces of singular moduli.

Since $\beta_0(j) = (\pi/3)720$ (see (24)), we immediately get the following special case of Corollary 1.3 which gives the limiting distribution of averages of short orbits of singular moduli.

**Corollary 1.4.** Let $(H_d)$ be a sequence of subgroups such that $|H_d| \gg |d|^\eta$ for some $\eta > \delta$. Then

$$\frac{1}{|H_d|} \sum_{\sigma \in H_d} j(\tau_{0,d})^\sigma - \frac{1}{h(d)} \sum_{Q \in \mathbb{Q}_d^\text{red}, y_Q > \frac{1}{\sqrt{d}} |d|^{-(1/2-\delta)}} C(\tau_{0,d}, Q) e(-\tau_Q) = 720 + O_{\epsilon}(|d|^{-(\delta-\delta)+\epsilon})$$

as $|d| \to \infty$.

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2. **From averages on short orbits to twisted traces**

Let $G$ be a finite abelian group and $H < G$ be a subgroup. Let $\hat{G}$ be the group of characters $\chi : G \to S^1$ and $H^1 < \hat{G}$ be the subgroup of characters $\chi$ which restrict to the identity on $H$. Given a function $f : G \to \mathbb{C}$, the Fourier transform $\hat{f} : \hat{G} \to \mathbb{C}$ is defined by

$$\hat{f}(\chi) = \sum_{\sigma \in G} \overline{\chi}(\sigma) f(\sigma).$$

The Poisson summation formula states that

$$\frac{1}{|H|} \sum_{\sigma \in H} f(\sigma) = \frac{1}{|G|} \sum_{\chi \in H^1} \hat{f}(\chi).$$

Let $\phi : \mathbb{H} \to \mathbb{C}$ be an $\text{SL}_2(\mathbb{Z})$-invariant function and $\tau \in \Lambda_d$ be a Heegner point. Define the evaluation map

$$e_{\phi,\tau} : G_d \to \mathbb{C}$$
by $e_{\phi,\tau}(\sigma) = \phi(\tau^\sigma)$. Then by the Poisson summation formula we have

$$\frac{1}{|H_d|} \sum_{\sigma \in H_d} e_{\phi,\tau}(\sigma) = \frac{1}{|G_d|} \sum_{\chi \in \tilde{H}_d} \hat{e}_{\phi,\tau_0}(\chi),$$

or equivalently,

$$\text{Av}_{H_d}(\phi, \tau) = \frac{1}{|G_d|} \sum_{\chi \in \tilde{H}_d} \text{Tr}_{\chi,d}(\phi, \tau), \quad (7)$$

where the twisted trace is defined by

$$\text{Tr}_{\chi,d}(\phi, \tau) := \sum_{\sigma \in G_d} \chi(\sigma) \phi(\tau^\sigma).$$

**Lemma 2.1.** Let $\tau = \tau_{[Q_j]} \in \Lambda_d$. Then

$$\text{Tr}_{\chi,d}(\phi, \tau) = \chi([Q_j])^{-1} \sum_{Q \in Q_d^{\text{red}}} \chi(Q) \phi(\tau_Q).$$

**Proof.** Write

$$Q_d/\text{SL}_2(\mathbb{Z}) = \{[Q_1], \ldots, [Q_{h(d)}]\}.$$ 

Set $\tau = \tau_{[Q_j]}$ for some fixed $j$. Then

$$\text{Tr}_{\chi,d}(\phi, \tau) = \sum_{i=1}^{h(d)} \chi([Q_i]) \phi(\tau_{[Q_i]}) = \sum_{i=1}^{h(d)} \chi([Q_i]) \phi(\tau_{[Q_i] \circ Q_j}).$$

Now, the form $Q_i \circ Q_j$ is $\text{SL}_2(\mathbb{Z})$-equivalent to a unique reduced form $Q_{ij}$, and $Q_i$ is $\text{SL}_2(\mathbb{Z})$-equivalent to $Q_{ij} \circ Q_j^{-1}$. Hence

$$\sum_{i=1}^{h(d)} \chi([Q_i]) \phi(\tau_{Q_i \circ Q_j}) = \sum_{i=1}^{h(d)} \chi([Q_{ij} \circ Q_j^{-1}]) \phi(\tau_{[Q_{ij}]})$$

$$= \chi([Q_j])^{-1} \sum_{i=1}^{h(d)} \chi([Q_{ij}]) \phi(\tau_{Q_{ij}})$$

$$= \chi([Q_j])^{-1} \sum_{Q \in Q_d^{\text{red}}} \chi(Q) \phi(\tau_Q). \quad \Box$$

### 3. Bounds for twisted traces of automorphic functions

Let $\mathcal{F}$ denote the standard fundamental domain for $\text{SL}_2(\mathbb{Z})$. Given two $\text{SL}_2(\mathbb{Z})$-invariant functions $\phi_1, \phi_2 : \mathbb{H} \to \mathbb{C}$, we define the Petersson inner product by

$$\langle \phi_1, \phi_2 \rangle := \int_{\mathcal{F}} \phi_1(z) \overline{\phi_2(z)} \, d\mu(z)$$

where

$$d\mu(z) := \frac{dx dy}{y^2}.$$
is the hyperbolic measure. The corresponding $L_2$-norm is given by

$$||\phi||_2 := \sqrt{\langle \phi, \phi \rangle} = \left( \int _{\mathcal{F}} |\phi(z)|^2 d\mu(z) \right)^{1/2}.$$  

Let $\mathcal{D}(\text{SL}_2(\mathbb{Z})\backslash \mathbb{H})$ denote the space of $\text{SL}_2(\mathbb{Z})$-invariant functions $\phi : \mathbb{H} \rightarrow \mathbb{C}$ such that $\phi$ and $\Delta \phi$ are both smooth and bounded, where

$$\Delta := -y^2 (\partial_x^2 + \partial_y^2)$$

is the hyperbolic Laplacian. For $A \in \mathbb{Z}^+$ we let $\Delta^A$ denote the composition of $\Delta$ with itself $A$-times.

**Proposition 3.1.** Let $\phi \in \mathcal{D}(\text{SL}_2(\mathbb{Z})\backslash \mathbb{H})$ and $\tau \in \Lambda_d$. Then

$$\text{Tr}_{\chi,d}(\phi, \tau) = C(\chi) \frac{3}{\pi} \langle \phi, 1 \rangle + O(\|\Delta^A \phi\|_2 |d|^{\delta_1/2+1/4+\epsilon/2}) + O(\|\Delta^A \phi\|_2 |d|^{\delta_2+1/4+\epsilon})$$

for any integer $A > \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\}$ where

$$C(\chi) := \sum _{\sigma \in G_d} \chi(\sigma).$$

**Proof.** Let $\{u_j\}_{j=1}^{\infty}$ be an orthonormal basis of Maass cusps forms for $\text{SL}_2(\mathbb{Z})$ with $\Delta$-eigenvalues $\lambda_j = 1/4 + t_j^2$. Define the non-holomorphic Eisenstein series

$$E(z, s) := \sum _{\gamma \in \Gamma_\infty \backslash \text{SL}_2(\mathbb{Z})} \text{Im}(\gamma z)^s, \quad \text{Re}(s) > 1$$

which is an eigenfunction for $\Delta$ with eigenvalue $s(1-s)$. We have the spectral expansion (see e.g. [9, Theorem 15.5])

$$\phi(z) = \frac{\langle \phi, 1 \rangle}{\text{vol}(\mathcal{F})} + \sum _{j=1}^{\infty} \langle \phi, u_j \rangle u_j(z) + \frac{1}{4\pi} \int _{\mathbb{R}} \langle \phi, E(\cdot, 1/2 + it) \rangle E(z, 1/2 + it) dt$$

which converges pointwise absolutely and uniformly on compact subsets of $\text{SL}_2(\mathbb{Z})\backslash \mathbb{H}$. Using $\text{vol}(\mathcal{F}) = \pi/3$, this gives

$$\text{Tr}_{\chi,d}(\phi, \tau) = C(\chi) \frac{3}{\pi} \langle \phi, 1 \rangle + \sum _{j=1}^{\infty} \langle \phi, u_j \rangle W_{\chi,d,j} + \frac{1}{4\pi} \int _{\mathbb{R}} \langle \phi, E(\cdot, 1/2 + it) \rangle W_{\chi,d}(t) dt$$

which converges pointwise absolutely and uniformly on compact subsets of $\text{SL}_2(\mathbb{Z})\backslash \mathbb{H}$.

Here,

$$W_{\chi,d,j} := \sum _{\sigma \in G_d} \chi(\sigma) u_j(\tau^\sigma)$$

and

$$W_{\chi,d}(t) := \sum _{\sigma \in G_d} \chi(\sigma) E(\tau^\sigma, 1/2 + it).$$

Now, by a formula of Waldspurger and Zhang [12, 14] (see also [6] and [11, Section 3]) we have

$$|W_{\chi,d,j}|^2 = \frac{\sqrt{|d|} L(\tilde{u}_j \otimes \chi, 1/2)}{2 L(\text{sym}^2 \tilde{u}_j, 1)}$$
where $\tilde{u}_j$ is the arithmetically normalized Maass form corresponding to $u_j$. Then by the Hoffstein/Lockhart bound \[7\]

\[ L(\text{sym}^2 \tilde{u}_j, 1) \gg \lambda_j^{-\epsilon} \]

and the bound (2)

\[ L(\tilde{u}_j \otimes \chi, 1/2) \ll \lambda_j^{B_1+\epsilon} |d|^{\delta_1+\epsilon} \]

we get

\[ W_{\chi,d,j} \ll \lambda_j^{B_1+\epsilon} |d|^{\delta_1/2+1/4+\epsilon/2}. \quad (9) \]

Similarly, by Gross/Zagier \[5\] we have

\[ W_{\chi,d}(t) = 2^{-(1/2+it)} |d|^{(1/2+it)/2} L(\chi, 1/2 + it) \frac{L(1/2 + it)}{\zeta(1 + 2it)}. \quad (10) \]

Then by the bound (3)

\[ L(\chi, 1/2 + it) \ll \epsilon (1/4 + t^2)^{B_2+\epsilon} |d|^{\delta_2+\epsilon} \]

and the standard bound

\[ \zeta(1 + 2it) \gg \epsilon (1/4 + t^2)^{-\epsilon} \quad (11) \]

we get

\[ W_{\chi,d}(t) \ll \epsilon (1/4 + t^2)^{B_2+2\epsilon} |d|^{\delta_2+1/4+\epsilon}. \quad (12) \]

By a repeated application of Stokes’ theorem (see e.g. \[8, Lemma 4.1\]), for any $A \in \mathbb{Z}^+$ we have

\[ \langle \phi, u_j \rangle = \lambda_j^{-A} \langle \Delta^A \phi, u_j \rangle \quad (13) \]

and

\[ \langle \phi, E(\cdot, 1/2 + it) \rangle = (1/4 + t^2)^{-A} \langle \Delta^A \phi, E(\cdot, 1/2 + it) \rangle. \quad (14) \]

Also, Parseval’s identity yields (see e.g. \[9, (15.17)\])

\[ \sum_{j=1}^{\infty} |\langle \Delta^A \phi, u_j \rangle|^2 + \frac{1}{4\pi} \int_{\mathbb{R}} |\langle \Delta^A \phi, E(\cdot, 1/2 + it) \rangle|^2 dt = ||\Delta^A \phi||^2. \quad (15) \]

Finally, by Weyl’s law

\[ |\{ t_j : |t_j| \leq T \}| \ll T^2 \]

and the bound $\lambda_j \geq 1/4$, summation by parts shows that the series

\[ \sum_{j=1}^{\infty} \frac{1}{\lambda_j^A} \quad (16) \]

converges for any integer $A' > 2$.

Then (13), (14), the Cauchy-Schwarz inequality, (15) and (16) give

\[ \sum_{j=1}^{\infty} \langle \phi, u_j \rangle \lambda_j^{B_1+\epsilon} \ll ||\Delta^A \phi||_2 \quad (17) \]
\[ \int_{\mathbb{R}} \langle \phi, E(\cdot, 1/2 + it) \rangle (1/4 + t^2)^{B_2 + 2\epsilon} dt \ll \| \Delta^A \phi \|_2 \]  

(18)

for any \( A > \max \{ B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon \} \).

The proposition now follows by combining (8), (9), (12), (17) and (18). \( \square \)

4. Fourier expansion of \( D^k f \)

Here we state the Fourier expansion of \( D^k f \). Recall that the Kloosterman sum is defined by

\[ S(a, b; c) := \sum_{d \equiv (c,d) = 1 (mod c)} e\left(\frac{ad + bd}{c}\right) \]

where \( \bar{d} \) is the multiplicative inverse of \( d \) (mod \( c \)). Also, let \( I_\nu \) denote the \( I \)-Bessel function of order \( \nu \).

By [10, Propositions 5.3 and 6.2], we have the following Fourier expansion.

**Proposition 4.1.** Let \( k \geq 0 \) be an integer and \( f \in M^!_{-2k}(1) \). Then

\[ D^k f(z) = \sum_{n=0}^{N_\infty} a(-n)e(-nz)c_k(n, y) + \sum_{n=1}^{\infty} B_k(n, y)e(nz), \]

where for \( 0 \leq n \leq N_\infty \)

\[ c_k(n, y) := \sum_{j=0}^{k} \frac{(-1)^j(k+j)!n^{k-j}}{(4\pi y)^j j!(k-j)!}, \]

and for \( n \geq 1 \),

\[ B_k(n, y) := \frac{2\pi}{\sqrt{n}} S_k(n) \sum_{j=0}^{k} \frac{(k+j)!}{(4\pi ny)^j j!(k-j)!}, \]

with

\[ S_k(n) := \sum_{m=1}^{N_\infty} a(-m)m^{k+1/2} \sum_{c>0} S(-m, n; c) I_{2k+1} \left( \frac{4\pi \sqrt{mn}}{c} \right). \]

5. Regularization of \( D^k f \)

In this section we recall the construction of a function which regularizes the function \( D^k f \) in the cusp at \( \infty \).

Let \( \phi_0 : \mathbb{R} \to [0, 1] \) be a \( C^\infty \) function such that

\[ \phi_0(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq 1. \end{cases} \]

(19)

Let \( 0 < \eta < 1 \). Define the Poincaré series

\[ f_{k, \eta}(z) := \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} g_{k, \eta}(\gamma z) \]

(20)
where
\[ g_{k,\eta}(z) := \sum_{m=0}^{N_{\infty}} a(-m)\psi_{m,k,\eta}(\text{Im}(z))e(-mz), \]
\[ \psi_{m,k,\eta}(y) := \phi_0\left(\frac{y - 2/\sqrt{3}}{\eta}\right)c_k(m, y). \]

Then define the regularized function
\[ f_{k,\eta}^{\text{reg}}(z) := \mathcal{D}^k f(z) - f_{k,\eta}(z). \] (21)

By [10, Proposition 7.1] we have the following result.

**Proposition 5.1.** For \( y \geq 2/\sqrt{3} + \eta \) we have
\[ f_{k,\eta}^{\text{reg}}(z) = \sum_{n=1}^{\infty} b_k(n, y) e(nz) \]
where \( b_k(n, y) := B_k(n, y) \) if \( k \geq 1 \) and \( b_k(n, y) := a(n) \) if \( k = 0 \).

6. **Regularized integrals**

For a fixed \( Y > 2/\sqrt{3} \), define the truncated fundamental domain
\[ \mathcal{F}_Y := \{ z \in \mathcal{F} : \text{Im}(z) \leq Y \}. \]

Then if \( f \in M^!_{-2k}(1) \) we define the regularized integral of \( \mathcal{D}^k f \) by
\[ \beta_k(f) = \int_{\text{reg}} \mathcal{D}^k f(z) d\mu := \lim_{Y \to \infty} \int_{\mathcal{F}_Y} \mathcal{D}^k f(z) d\mu. \] (22)

By [10, Lemma 10.3], this limit always exists and
\[ \langle f_{k,\eta}^{\text{reg}}, 1 \rangle = \beta_k(f). \] (23)

Finally, by [10, Proposition 10.4] we have
\[ \beta_0(f) = \frac{\pi}{3} \left( a(0) - 24 \sum_{n=1}^{N_{\infty}} a(-n)\sigma_1(n) \right) \]
where \( \sigma_1(n) \) is the sum of all positive divisors of \( n \). In particular, if \( f = j \) is the modular \( j \)-function, then
\[ \beta_0(j) = \frac{\pi}{3} 720. \] (24)

7. **Proof of Theorem 1.1**

We will deduce Theorem 1.1 from the following result.
Proposition 7.1. Let $\tau_{0,d} = \tau_{[Q_{0,d}]} \in \Lambda_d$. Then

$$\text{Tr}_{\chi,d}(D^k f, \tau_{0,d}) = \chi([Q_{0,d}])^{-1} \sum_{Q \in Q_d^\text{red}} \chi(Q) \sum_{m=0}^{N} a(-m)c_k(m,y_Q) e(-m\tau_Q) + C(\chi) \frac{3}{\pi} \beta_k(f)$$

$$+ O_\epsilon(\eta^{-2A}|d|^{\frac{1}{5}/2+1/4+\epsilon}) + O_\epsilon(\eta^{-2A}|d|^{\frac{1}{5}+1/4+\epsilon/2})$$

$$+ O(\eta|d|^{1/2+\epsilon}) + O_\epsilon(\eta^{-2A}|d|^{\frac{1}{5}+1/4+\epsilon})$$

for any integers $A, A' > 0$ with $A > \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\}$ and $A' > B_3 + B_4 + 3/2 + 3\epsilon$.

Proof. By (21) we have

$$\text{Tr}_{\chi,d}(D^k f, \tau_{0,d}) = \text{Tr}_{\chi,d}(f_{k,\eta}^\text{reg}, \tau_{0,d}) + \text{Tr}_{\chi,d}(f_{k,\eta}, \tau_{0,d}).$$

By Proposition 5.1 we have $f_{k,\eta}^\text{reg} \in D(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H})$. Hence by Proposition 3.1 and (23) we get

$$\text{Tr}_{\chi,d}(f_{k,\eta}^\text{reg}, \tau_{0,d}) = C(\chi) \frac{3}{\pi} \beta_k(f) + O_\epsilon(\eta^{-2A}|d|^{\frac{1}{5}/2+1/4+\epsilon}) + O_\epsilon(\eta^{-2A}|d|^{\frac{1}{5}+1/4+\epsilon/2})$$

for any integer $A > \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\}$.

By Lemma 2.1 we have

$$\text{Tr}_{\chi,d}(f_{k,\eta}, \tau_{0,d}) = \chi([Q_{0,d}])^{-1} \sum_{Q \in Q_d^\text{red}} \chi(Q) f_{k,\eta}(\tau_Q).$$

A straightforward modification of [10, Lemma 9.1] yields the decomposition

$$\sum_{Q \in Q_d^\text{red}} \chi(Q) f_{k,\eta}(\tau_Q) = T_{\chi,d,1} + T_{\chi,d,2}$$

where

$$T_{\chi,d,1} := \sum_{Q \in Q_d^\text{red}} \chi(Q) \sum_{m=0}^{N} a(-m)c_k(m,y_Q) e(-m\tau_Q)$$

$$\text{with } y_Q > \frac{2\eta}{\sqrt{3}} + \eta$$

$$T_{\chi,d,2} := \sum_{Q \in Q_d^\text{red}} \chi(Q) \sum_{m=0}^{N} a(-m)c_k(m,y_Q) e(-m\tau_Q).$$

$$\text{with } \sqrt{2}/3 < y_Q \leq \frac{2}{\sqrt{3}} + \eta$$

Since $|\chi([Q])| = 1$ for any $[Q] \in Q_d/\text{SL}_2(\mathbb{Z})$, an estimate gives

$$T_{\chi,d,2} \ll \Lambda(d, \eta)$$

where

$$\Lambda(d, \eta) := |\{Q \in Q_d^\text{red} : \sqrt{2}/3 < y_Q \leq 2/\sqrt{3} + \eta\}|.$$
We next bound \( \Lambda(d, \eta) \) along the lines of [10, Lemma 9.2]. Let \( \phi_\eta : \mathbb{R} \to [0, 1] \) be a smooth function which is supported on \( (2/\sqrt{3} - \eta, 2/\sqrt{3} + 2\eta) \), which equals 1 on \( [2/\sqrt{3}, 2/\sqrt{3} + \eta] \), and which satisfies
\[
\phi_\eta^{(l)} \ll \eta^{-l}, \quad l = 0, 1, 2, \ldots. \tag{25}
\]
Define the Poincaré series
\[
P_\eta(z) := \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \phi_\eta(\im(\gamma z)).
\]
Then by construction we get
\[
|\Lambda(d, \eta)| \leq \sum_{Q \in \mathcal{Q}_d} P_\eta(\tau_Q).
\]
By [8, (7.12)] we have
\[
P_\eta(z) = \frac{3}{\pi} \widehat{\phi}_\eta(1) + \frac{1}{2\pi i} \int_{\mathbb{R}} \widehat{\phi}_\eta(1/2 + it) E(z, 1/2 + it) dt
\]
where
\[
\hat{\phi}_\eta(s) := \int_0^\infty \phi_\eta(u) u^{-(s+1)} du.
\]
Thus
\[
\sum_{Q \in \mathcal{Q}_d} P_\eta(\tau_Q) = \frac{3}{\pi} \hat{\phi}_\eta(1) h(d) + \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{\phi}_\eta(1/2 + it) W_{1,d}(t) dt.
\]
An estimate gives
\[
\hat{\phi}_\eta(1) \ll \eta.
\]
Further, by (10) and the factorization
\[
\zeta_{\mathbb{Q}(\sqrt{d})}(s) = \zeta(s) L(\chi_d, s)
\]
we have
\[
W_{1,d}(t) = 2^{-(1/2+it)} |d|^{(1/2+it)/2} \frac{\zeta(1/2 + it)}{\zeta(1 + 2it)} L(\chi_d, 1/2 + it).
\]
Hence the bounds (11), (5) and (4) yield
\[
W_{1,d}(t) \ll \epsilon \left(1/4 + t^2\right)^{B_3 + B_4 + 3\epsilon} |d|^{\delta_3 + 1/4 + \epsilon}.
\]
It follows that
\[
\sum_{Q \in \mathcal{Q}_d} P_\eta(\tau_Q) = O(\eta h(d)) + O(\epsilon c(\eta)|d|^{\delta_3 + 1/4 + \epsilon})
\]
where
\[
c(\eta) := \int_0^\infty |\hat{\phi}_\eta(1/2 + it)|(1/4 + t^2)^{B_3 + B_4 + 3\epsilon} dt.
\]
Integrate by parts \( A^\prime \)-times and use the bound (25) to obtain
\[
\hat{\phi}_\eta(1/2 + it) \ll \frac{\eta^{-(A^\prime-1)}}{(1/4 + t^2)^{A^\prime-1}}.
\]
Hence
\[ c(\eta) \ll \eta^{-(A'-1)} \]
for \( A' > B_3 + B_4 + 3/2 + 3\epsilon \). We have shown
\[ \Lambda(d, \eta) = O(\eta h(d)) + O(\eta^{-(A'-1)}|d|^\delta_1 + 1/4 + \epsilon). \]
An inspection of [10, Lemma 9.3] gives
\[ \| \Delta^A f_{k,\eta}^\text{reg} \|_2 \ll \eta^{-2A}. \]
Further, we have the bound
\[ h(d) \ll |d|^{1/2 + \epsilon}. \]
Putting things together, we obtain the result.

\[ \square \]

**Proof of Theorem 1.1.** By Proposition 7.1, (7) and orthogonality, we have
\[ \text{Av}_{H_d}(D^k f, \tau_{0,d}) = \frac{1}{h(d)} \sum_{Q \in \mathcal{Q}_{\text{red}}^{\text{reg}}} C(\tau_{0,d}, Q) \sum_{m=0}^{N_\infty} a(-m)c_k(m, y_Q) e(-m\tau_Q) + \frac{3}{\pi} \beta_k(f) + E(\delta_1, \delta_2, \delta_3, A, A') \]
where
\[ C(\tau_{0,d}, Q) := \sum_{\chi \in H_d} \chi([Q_{\tau_{0,d}}^{-1} \circ Q]), \]
and the error term is
\[ E(\delta_1, \delta_2, \delta_3, A, A') := O(\|H_d\|^{-1} \eta^{-2A}|d|^{\delta_1/2 + 1/4 + \epsilon}) + O(\|H_d\|^{-1} \eta^{-2A}|d|^{\delta_2 + 1/4 + \epsilon/2}) + O(\|H_d\|^{-1} \eta^{-(A'-1)}|d|^{\delta_3 + 1/4 + \epsilon}) \]
for any \( A > \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\} \) and \( A' > B_3 + B_4 + 3/2 + 3\epsilon \).

Let \( \eta = |d|^{-b} \), where \( b \) will be chosen to minimize the error term. We have
\[ E(\delta_1, \delta_2, \delta_3, A, A') = O(\|H_d\|^{-1}|d|^{\delta + \epsilon}), \]
where
\[ \delta := \delta(\delta_1, \delta_2, \delta_3, A, A') = \max \left\{ \frac{\delta_1}{2} + \frac{1}{4} + 2Ab, \frac{1}{4} + 2Ab, \frac{1}{2} - b, \delta_3 + \frac{1}{4} + (A' - 1)b \right\}. \]

As \( b \) varies, \( \delta \) is minimized when
\[ \delta = \delta(\delta_1, \delta_2, \delta_3, A, A') = \left\{ \begin{array}{ll} p_1, & \text{if } p_1 \leq p_2 \text{ and } p_1 \leq p_3 \\ p_2, & \text{if } p_2 \leq p_1 \text{ and } p_2 \leq p_3 \\ p_3, & \text{if } p_3 \leq p_1 \text{ and } p_3 \leq p_2 \end{array} \right. \]
where
\[
p_1 := \frac{1 - 2\delta_1}{4(2A + 1)},
\]
\[
p_2 := \frac{1 - 4\delta_2}{4(2A + 1)},
\]
\[
p_3 := \frac{1 - 4\delta_3}{4A'},
\]
Moreover, the minimal value of \(\delta\) is \(\frac{1}{2} - b_0\).

Setting \(A = \lceil \max\{B_1/2 + 1 + \epsilon, \bar{B}_2 + 1/4 + 2\epsilon\} \rceil + 1\) and \(A' = \lceil B_3 + B_4 + 3/2 + 3\epsilon \rceil + 1\) yields the result.

\[\square\]

REFERENCES