Algebraic properties of values of newform Dedekind sums

Mitch Majure

REU, Texas A&M University, College Station, TX

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Section 1

1 Preliminaries

2 Defining Dedekind sums

3 Hecke Operators

4 Galois Action and Structure
A Dirichlet character (modulo q) $\chi : \mathbb{Z} \to \mathbb{C}^\times$ is a mapping with the following properties:

- $\chi(ab) = \chi(a)\chi(b)$.
- $\chi(a) = \begin{cases} 
0 & \text{gcd}(a, q) \neq 1 \\
\neq 0 & \text{gcd}(a, q) = 1.
\end{cases}$
- $\chi(a \pm q) = \chi(a)$.

These properties imply that when $(a, q) = 1$, $\chi(a)$ is a $\phi(q)^{th}$ roots of unity.
Recall the following definitions:

**Definition**

\[
SL_2(\mathbb{Z}) = \{ \gamma \in M_2(\mathbb{Z}) \mid \det(\gamma) = 1 \}
\]

\[
\Gamma_0(N) = \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \pmod{N} \}
\]

\[
\Gamma_1(N) = \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod{N} \}
\]

\[
\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{N} \}
\]

**Remark**

\(\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})\).

\(\Gamma(N)\) is the kernel of the reduction map \(SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/N\mathbb{Z})\).
A modular form $f$ of weight $k$ is a function on $\mathbb{H}$ that has a symmetry with the group action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ by linear fractional transformations. Specifically:

- $f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} z \right) = f\left(\frac{az+b}{cz+b} \right) = (cz+d)^k f(z)$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.
- $f$ is holomorphic/complex analytic.
- $f(z)$ is bounded as $\text{Im}(z) \to \infty$.

We can extend this concept to what are called automorphic forms by relaxing the holomorphicity requirement and including an automorphy factor $\epsilon$ in the symmetry:

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} z \right) = f\left(\frac{az+b}{cz+b} \right) = \epsilon(a, b, c, d)(cz+d)^k f(z).$$
Section 2

1 Preliminaries
2 Defining Dedekind sums
3 Hecke Operators
4 Galois Action and Structure
Eisenstein Series

Eisenstein series of weight 0 attached to Dirichlet characters $\chi_1$, $\chi_2$ (modulo $q_1$, $q_2$ respectively), $E_{\chi_1,\chi_2}(z, s)$.

Automorphic form on the congruence subgroup $\Gamma_0(q_1q_2)$ with central character $\psi = \chi_1 \chi_2$. This means $E_{\chi_1,\chi_2}(\gamma z, s) = \psi(\gamma) E_{\chi_1,\chi_2}(z, s)$ for $\gamma \in \Gamma_0(q_1q_2)$.

Note $\psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \psi(d)$.

Eigenfunction of all Hecke operators $T_n$ with eigen value $\lambda_{\chi_1,\chi_2}(n, s)$.

$E^*_{\chi_1,\chi_2}(z, s)$ (the completed Eisenstein series) at $s = 1$ decomposes into holomorphic and anti-holomorphic parts $f_{\chi_1,\chi_2}(z) + \chi_2(-1) f_{\chi_1,\chi_2}(z)$. 
Eisenstein Series

- Eisenstein series of weight 0 attached to Dirichlet characters \( \chi_1, \chi_2 \) (modulo \( q_1, q_2 \) respectively), \( E_{\chi_1, \chi_2}(z, s) \).
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  (Note $\psi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \psi(d)$.)
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- Eisenstein series of weight 0 attached to Dirichlet characters \( \chi_1, \chi_2 \) (modulo \( q_1, q_2 \) respectively), \( E_{\chi_1, \chi_2}(z, s) \).
- Automorphic form on the congruence subgroup \( \Gamma_0(q_1q_2) \) with central character \( \psi = \chi_1 \overline{\chi_2} \). This means \( E_{\chi_1, \chi_2}(\gamma z, s) = \psi(\gamma) E_{\chi_1, \chi_2}(z, s) \) for \( \gamma \in \Gamma_0(q_1q_2) \).
  \[ \left( \text{Note } \psi\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \psi(d) \right). \]
- Eigenfunction of all Hecke operators \( T_n \) with eigen value \( \lambda_{\chi_1, \chi_2}(n, s) \).
- \( E^*_{\chi_1, \chi_2}(z, s) \) (the completed Eisenstein series) at \( s = 1 \) decomposes into holomorphic and anti-holomorphic parts \( f_{\chi_1, \chi_2}(z) + \chi_2(-1) \overline{f_{\chi_1, \chi_2}(z)} \).
Original Definition of $S_{\chi_1 \chi_2}$

The holomorphic part has the following Fourier expansion:

$$f_{\chi_1, \chi_2}(z) = \sum_{n=1}^{\infty} \lambda_{\chi_1, \chi_2}(n, 1) \sqrt{n} \exp(2\pi i nz).$$

Where

$$\lambda_{\chi_1, \chi_2}(n, s) = \chi_2(sgn(n)) \left| \frac{n}{a} \right| \chi_1(a) \chi_2(b) b a^{-s-\frac{1}{2}}.$$
Holomorphic part has the following fourier expansion:

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\[ \lambda_{\chi_1, \chi_2}(n, s) = \chi_2(\text{sgn}(n)) \sum_{ad=|n|} \chi_1(a)\overline{\chi_2(b)} \left( \frac{b}{a} \right)^{s-\frac{1}{2}}. \]
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- The first definition of the newform Dedekind sum is as follows:

**Definition**

For primitive $\chi_1, \chi_2 \mod q_1, q_2$ where $\chi_1\chi_2(-1) = 1$, and $\gamma \in \Gamma_0(q_1q_2)$

$$S_{\chi_1\chi_2}(\gamma) = \frac{\tau(\overline{\chi_1})}{\pi i} (f_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) f_{\chi_1,\chi_2}(z))$$
Theorem

\[
\chi_1 \chi_2(\gamma) = \chi_1 \chi_2(a, c) = X_j \mod c X_n \mod q_1 \chi_2(j) \chi_1(n) B_1(jc) + ajc \]

where \(B_1(x) = (0, \text{if } x \in \mathbb{Z}, x - \lfloor x \rfloor - \frac{1}{2})\) otherwise.

From the above theorem, we see the sum always lies in \(\mathbb{Q}(\zeta_n)\) (\(\mathbb{Q}\) adjoined with some root of unity \(\zeta_n\)). \(\chi_1 \chi_2\) is never trivial.

\[
\chi_1 \chi_2(\gamma_1 \gamma_2) = \chi_1 \chi_2(\gamma_1) + \psi(\gamma_1) \chi_1 \chi_2(\gamma_2).
\]

\(\chi_1 \chi_2\) is a homomorphism when \(\psi = 1\), while \(\chi_1 \chi_2|_{\Gamma_1(q_1 q_2)}\) is always a homomorphism.
Properties

Theorem

\[ S_{\chi_1 \chi_2}(\gamma) = S_{\chi_1 \chi_2}(a, c) = \sum_{j \mod c} \sum_{n \mod q_1} \overline{\chi_2}(j) \overline{\chi_1}(n) B_1 \left( \frac{j}{c} \right) B_1 \left( \frac{n}{q_1} + \frac{aj}{c} \right) \]

where

\[ B_1(x) = \begin{cases} 
0, & \text{if } x \in \mathbb{Z} \\
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- \( S_{\chi_1 \chi_2} \) is never trivial.
Theorem

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- From the above theorem, we see the sum always lies in \( \mathbb{Q}(\zeta_n) \) (\( \mathbb{Q} \) adjoined with some root of unity \( \zeta_n \)).
- \( S_{\chi_1 \chi_2} \) is never trivial.
- \( S_{\chi_1 \chi_2}(\gamma_1 \gamma_2) = S_{\chi_1 \chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1 \chi_2}(\gamma_2) \).
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\[ S_{\chi_1 \chi_2}(\gamma) = S_{\chi_1 \chi_2}(a, c) = \sum_{j \mod c} \sum_{n \mod q_1} \chi_2(j) \chi_1(n) B_1 \left( \frac{j}{c} \right) B_1 \left( \frac{n}{q_1} + \frac{aj}{c} \right) \]

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- \( S_{\chi_1 \chi_2} \) is a homomorphism when \( \psi = 1 \), while \( S_{\chi_1 \chi_2}|_{\Gamma_1(q_1q_2)} \) is always a homomorphism.
Section 3

1. Preliminaries
2. Defining Dedekind sums
3. Hecke Operators
4. Galois Action and Structure
The Hecke operator $T_n$

**Definition**

The weight 0 Hecke operator on automorphic forms with central character $\psi$ is

$$T_n = \frac{1}{\sqrt{n}} \sum_{ad=n} \psi(n) \sum_{b \pmod{d}} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$
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$$T_n = \frac{1}{\sqrt{n}} \sum_{ad=n} \psi(n) \sum_{b \equiv b (\text{mod } d)} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$  

As noted, $E_{\chi_1, \chi_2}(z, s)$ is an eigen function for this family of commuting linear operators with eigenvalue $\lambda_{\chi_1, \chi_2}(n, s)$. When we specialize to $s = 1$ we deduce that $f_{\chi_1, \chi_2}(z)$ is as well:

$$T_n f_{\chi_1, \chi_2}(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \psi(n) \sum_{b \equiv b (\text{mod } d)} f_{\chi_1, \chi_2}(\frac{az + b}{d})$$

$$= \lambda_{\chi_1, \chi_2}(n, 1)f_{\chi_1, \chi_2}(z).$$
This gives an easy definition of $T_n S_{\chi_1 \chi_2}$:

$$T_n S_{\chi_1 \chi_2} (\gamma) = \tau(\chi_1) \pi_i (T_n f_{\chi_1, \chi_2}(\gamma z)) - \psi(\gamma) T_n f_{\chi_1, \chi_2}(z) = \lambda_{\chi_1, \chi_2}(n, 1) S_{\chi_1 \chi_2}(\gamma).$$

We can then use the Fourier expansion of $f_{\chi_1, \chi_2}(z)$ and the $z$-independence of $S_{\chi_1 \chi_2}(\gamma)$ to get the following identity:

**Theorem**

For $h, k, n \in \mathbb{Z}$, $q_1 | k$, and $n, k > 0$

$$\sum_{a=1}^{\sqrt{n}} \alpha(a) X_{b \pmod{d}} S_{\chi_1 \chi_2}(ah + bk, dk) = \lambda_{\chi_1, \chi_2}(n, 1) S_{\chi_1 \chi_2}(h, k).$$
This gives an easy definition of $T_n S_{\chi_1\chi_2}$:

$$T_n S_{\chi_1\chi_2}(\gamma) = \frac{\tau(\chi_1)}{\pi i} (T_n f_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) T_n f_{\chi_1,\chi_2}(z))$$

$$= \lambda_{\chi_1,\chi_2}(n, 1) S_{\chi_1\chi_2}(\gamma).$$
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T_n S_{\chi_1 \chi_2}(\gamma) = \frac{\tau(\chi_1)}{\pi i} (T_n f_{\chi_1, \chi_2}(\gamma z) - \psi(\gamma)T_n f_{\chi_1, \chi_2}(z))
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**Theorem**

For $h, k, n \in \mathbb{Z}$, $q_1 q_2 | k$, and $n, k > 0$

$$
\frac{1}{\sqrt{n}} \sum_{ad=n} \chi_1 \overline{\chi_2}(a) \sum_{b \pmod d} S_{\chi_1 \chi_2}(ah + bk, dk) = \lambda_{\chi_1, \chi_2}(n, 1) S_{\chi_1 \chi_2}(h, k).
$$
This is a generalization of the classical case due to M. Knopp:

**Theorem**

For $h, k, n \in \mathbb{Z}$, $k, n > 0$

$$\sum_{ad=n} \sum_{b \equiv 0 \pmod{d}} s(ah + bk, dk) = \sigma(n)s(h, k), \quad \sigma(n) = \sum_{d|n} d.$$

More importantly, the fact that the Dedekind sums are eigenvectors of a family of commuting linear operators means they are linearly independent.
Section 4

1 Preliminaries

2 Defining Dedekind sums

3 Hecke Operators

4 Galois Action and Structure
Definition of Galois action

Let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Then,

$$\sigma S_{\chi_1 \chi_2}(\gamma) = \sigma \left( \sum_{j \mod c} \sum_{n \mod q_1} \chi_2(j) \chi_1(n) B_1 \left( \frac{j}{c} \right) B_1 \left( \frac{n}{q_1} + \frac{aj}{c} \right) \right)$$

$$= \sum_{j \mod c} \sum_{n \mod q_1} \overline{\chi_2}(j) \overline{\chi_1}(n) B_1 \left( \frac{j}{c} \right) B_1 \left( \frac{n}{q_1} + \frac{aj}{c} \right)$$

$$= S_{\chi_1^{\sigma}, \chi_2^{\sigma}}(\gamma)$$
The Number Field Containing $S_{\chi_1 \chi_2}(\Gamma_0(q_1q_2))$
The Number Field Containing $S_{\chi_1\chi_2}(\Gamma_0(q_1q_2))$

**Theorem**

The Dedekind sum $S_{\chi_1\chi_2}$ takes values in the number field $F$ if and only if $\chi_1$ and $\chi_2$ take values in $F$.
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The backward direction of this theorem comes directly from the finite sum definition. For the other, we choose any $\sigma \in Gal(\overline{Q}/F)$ and must have

$$S_{\chi_1\chi_2} = S_{\chi_1^\sigma,\chi_2^\sigma}.$$

The linear independence of the sums then gives $\chi_1 = \chi_1^\sigma$ and $\chi_2 = \chi_2^\sigma$, proving the result.
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$S_{\chi_1 \chi_2}$ takes rational values if and only if $\chi_1$ and $\chi_2$ are rational characters.
Structure of $\Gamma_1(q_1q_2)/K^1_{\chi_1,\chi_2}$

We use the following definitions:

- $K_{\chi_1,\chi_2} = \{ \gamma \in \Gamma_0(q_1q_2) | S_{\chi_1\chi_2}(\gamma) = 0 \}$
- $K^1_{\chi_1,\chi_2} = \Gamma_1(q_1q_2) \cap K_{\chi_1,\chi_2}$
- $F$ is the smallest number field over $\mathbb{Q}$ in which $S_{\chi_1\chi_2}$ takes values.
Structure of $\Gamma_1(q_1q_2)/K_{\chi_1,\chi_2}^1$

We use the following definitions:

- $K_{\chi_1,\chi_2} = \{ \gamma \in \Gamma_0(q_1q_2) | S_{\chi_1\chi_2}(\gamma) = 0 \}$
- $K_{\chi_1,\chi_2}^1 = \Gamma_1(q_1q_2) \cap K_{\chi_1,\chi_2}$
- $F$ is the smallest number field over $\mathbb{Q}$ in which $S_{\chi_1\chi_2}$ takes values.

We will investigate $\Gamma_1(q_1q_2)/K_{\chi_1,\chi_2}^1 \cong S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$ because $S_{\chi_1\chi_2}|_{\Gamma_1(q_1q_2)}$ is a homomorphism, and we will show its rank is equal to the degree of $F$ over $\mathbb{Q}$. These arguments are carried over without changes to $\Gamma_0(q_1q_2)/K_{\chi_1,\chi_2}$ in the case of $\chi_1\overline{\chi_2} = 1$. 
Lemma (Consequence of) Schrier’s lemma

Every finite index subgroup of a finitely generated group is finitely generated.

Since $\Gamma_1(q_1q_2)$ is of finite index in $\SL_2(\mathbb{Z}) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$, it must also have a set of finite generators $\{\gamma_j\}_{j=1}^r$. Then $\{S_{\chi_1\chi_2}(\gamma_j)\}_{j=1}^r$ generates $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$.

We also know $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2)) \subset (\mathbb{C}, +)$ implies $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$ is torsion free.

Combining these two facts shows $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$ is a free abelian group by the structure theorem of abelian groups.
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- We also know \( S_{\chi_1\chi_2}(\Gamma_1(q_1q_2)) \subset (\mathbb{C}, +) \) implies \( S_{\chi_1\chi_2}(\Gamma_1(q_1q_2)) \) is torsion free.
- Combining these two facts shows \( S_{\chi_1\chi_2}(\Gamma_1(q_1q_2)) \) is a free abelian group by the structure theorem of abelian groups.
Bounding the Rank

Lemma

The rank of $\chi_1 \chi_2 (\Gamma_1(q_1 q_2))$ is bounded above by the degree of $F/\mathbb{Q}$.

Proof sketch:

Since $F$ is the fraction field of its algebraic integers ($\mathcal{O}_F$), we may "bound" the denominators of $S_{\chi_1 \chi_2} (\gamma_j)$ by some $d \in F$. Then $S_{\chi_1 \chi_2} (\Gamma_1(q_1 q_2)) \subset 1/d \mathcal{O}_F$ (and its fractional ideals) are free abelian groups of rank $[F:\mathbb{Q}]$. 
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Proof sketch:

- Since $F$ is the fraction field of its algebraic integers ($\mathcal{O}_F$), we may "bound" the denominators of $S_{\chi_1 \chi_2}(\gamma_j)$ by some $d \in F$.
- Then $S_{\chi_1 \chi_2}(\Gamma_1(q_1 q_2)) \subset \frac{1}{d} \mathcal{O}_F$
Bounding the Rank

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The rank of $S_{\chi_1 \chi_2}(\Gamma_1(q_1q_2))$ is bounded above by the degree of $F/\mathbb{Q}$

Proof sketch:

- Since $F$ is the fraction field of its algebraic integers ($\mathcal{O}_F$), we may "bound" the denominators of $S_{\chi_1 \chi_2}(\gamma_j)$ by some $d \in F$.
- Then $S_{\chi_1 \chi_2}(\Gamma_1(q_1q_2)) \subset \frac{1}{d} \mathcal{O}_F$
- $\mathcal{O}_F$ (and its fractional ideals) are free abelian groups of rank $[F : \mathbb{Q}]$. 
Bounding the Rank

Lemma

The rank of $S_{\chi_1 \chi_2}(\Gamma_1(q_1 q_2))$ is bounded below by the degree of $F/\mathbb{Q}$.

Proof sketch:
First, nontriviality of $S_{\chi_1 \chi_2}(\Gamma_1(q_1 q_2))$ means the rank is at least one.
Suppose $S_{\chi_1 \chi_2}(\Gamma_1(q_1 q_2)) = \sum_{i=1}^{d} \alpha_i \mathbb{Z}$ for $\alpha_i \in F$, $d < [F:\mathbb{Q}] = n$.
Then consider the $n$ distinct Dedekind sums $S_{\chi_{\sigma_j 1} \chi_{\sigma_j 2}}$ for $\sigma_j \in \text{Gal}(F/\mathbb{Q})$.
Then we can construct a $d \times n$ matrix $(\alpha_{\sigma_j i})_{ij}$, that, by its dimension, has nontrivial kernel, contradicting the linear independence of the Dedekind sums.
Lemma

The rank of $S_{\chi_1\chi_2}(\Gamma_1(q_1q_2))$ is bounded below by the degree of $F/\mathbb{Q}$

Proof sketch:
Bounding the Rank

Lemma

The rank of $S_{\chi_1 \chi_2}(\Gamma_1(q_1 q_2))$ is bounded below by the degree of $F/\mathbb{Q}$

Proof sketch:
- First, nontriviality of $S_{\chi_1 \chi_2}$ means the rank is at least one.
Bounding the Rank

Lemma

The rank of $S_{\chi_1 \chi_2}(\Gamma_1(q_1q_2))$ is bounded below by the degree of $F/\mathbb{Q}$

Proof sketch:
- First, nontriviality of $S_{\chi_1 \chi_2}$ means the rank is at least one.
- Suppose $S_{\chi_1 \chi_2}(\Gamma_1(q_1q_2)) = \bigoplus_{i=1}^{d} \alpha_i \mathbb{Z}$ for $\alpha_i \in F$, $d < [F : \mathbb{Q}] = n$. 
Bounding the Rank

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The rank of $S_{\chi_1 \chi_2}(\Gamma_1(q_1 q_2))$ is bounded below by the degree of $F/\mathbb{Q}$

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