Counting and Finding Real Roots of Univariate Trinomials

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where \( c \neq 0 \), \( 0 < m < n \), and \( \gcd(m, n) = 1 \) and defining

\[ r_{m,n} := \left| \frac{n}{m^n (n - m)^{(n-m)/n}} \right| \]

there are various cases when we compare the coefficient \( c \) to \( r_{m,n} \).
When $c > r_{m,n}$, there are two positive roots.
Case 1

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- The smaller root of $f$ is given by

$$x_{\text{low}}(c) = \frac{1}{c^m} \left[ 1 + \sum_{k=1}^{\infty} \left( \frac{1}{km^k} \cdot \prod_{j=1}^{k-1} \frac{1 + kn - jm}{j} \right) \frac{1}{c^m} \right]$$
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- The larger root of $f$ is given by

$$x_{\text{hi}}(c) = c^{\frac{1}{(n-m)}} \left[ 1 - \sum_{k=1}^{\infty} \left( \frac{1}{k(n-m)^k} \cdot \prod_{j=1}^{k-1} \frac{km + j(n - m) - 1}{j} \right) \frac{1}{c^{\frac{kn}{(n-m)}}} \right]$$
Case 2

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- The following series converges near the root of \( f \)

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x_{\text{mid}}(c) = (-1)^{\frac{1}{n}} \left[ 1 + \sum_{k=1}^{\infty} \left( \frac{1}{km^k} \cdot \prod_{j=1}^{k-1} \frac{1 + km - jn}{j} \right) \left( -1 \right)^{\frac{m-n}{n}} c^k \right]
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However, how can you solve without knowing how many roots there are?
Let us consider

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with 

- \( c_1, c_2, c_3 \neq 0 \)
- \( 0 < a_2 < a_3 \)
- \( c_3 > 0 \)
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By examining the sign of the coefficients, we can determine the number of roots \( f \) will have

1. \( c_1, c_2 > 0 \) \( \Rightarrow \) 0 roots
2. \( c_1, c_2 < 0 \) \( \Rightarrow \) 1 root
3. \( c_1 < 0 \) and \( c_2 > 0 \) \( \Rightarrow \) 1 root
Root Counting

When \( c_1 > 0 \) and \( c_2 < 0 \), \( f \) can have 0, 1, or 2 roots. By evaluating the modified A-discriminant:

\[
Ξ_A = \left( \frac{c_1}{a_3 - a_2} \right)^{a_3-a_2} \left( \frac{c_2}{-a_3} \right)^{-a_3} \left( \frac{c_3}{a_2} \right)^{a_2} - 1
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if this value is:

1. $> 0 \Rightarrow 2$ roots
2. $= 0 \Rightarrow 1$ root
3. $< 0 \Rightarrow 0$ roots
Baker’s Theorem

This evaluation becomes complex with large coefficients, which is why we take advantage of an application of Baker’s Theorem.
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**Baker’s Theorem (1966)**

If $\alpha_i \in \mathbb{Q}_+$, $b_i \in \mathbb{Z}$ with $\log A_i := \max\{ h(\alpha_i), |\log(\alpha_i)|, 0.16 \}$, $B := \max\{|b_i|\}$, then

$$\sum_{i=1}^{m} b_i \log(\alpha_i) \neq 0 \Rightarrow$$

$$\log \left| \sum_{i=1}^{m} b_i \log(\alpha_i) \right| > -1.4 \cdot m^{4.5} \cdot 30^m \cdot (1 + \log(B)) \prod_{i=1}^{m} \log(A_i)$$
Approximating Logarithms

Lemma
Given any $x \in \mathbb{Q}_+$ of height $h$, and $\ell \in \mathbb{N}$ with $\ell \geq h$ we can compute $\lfloor \log_2 \max\{1, \log(x)\} \rfloor$ and the $\ell$ most significant bits of $\log(x)$ in time $O(\ell \log^2(\ell))$. 
Example

\[ f(x) = c_1 + c_2 x^2 + c_3 x^{13} \]
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Evaluating the modified A-discriminant and taking log

\[ 11 \log \left( \frac{c_1}{11} \right) - 13 \log \left( \frac{c_2}{-13} \right) + 2 \log \left( \frac{c_3}{2} \right) \]
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We can approximate using

\[ 11 L_1 - 13 L_2 + 2 L_3 \]

where \( L_i \approx \log(\alpha_i) \) up to error \(< \frac{1}{10} (-1.4 \cdot m^{4.5} 30^{m+3} (1 + \log(B)) \prod_{i=1}^{m} \log(A_i)) \)
Case 3

Consider the trinomial

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- However these series do not produce 2 correct decimal places of until at least 20,000 terms are used.
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So, we need a new series to solve this case.
When $|c| \approx r_{m,n}$ the following pair of series give the roots of $f$
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\[ x_{\text{sing}}^{\pm} = \zeta \sum_{k=0}^{\infty} \frac{\gamma_k}{(\pm \sqrt{(n-m)r})^k} (c - r)^{k/2} \]

where $\zeta = \left(\frac{m}{n-m}\right)^{1/n}$ and $\gamma_k \in \mathbb{Q}[m, n]$
Singular Series

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We can find $\gamma_k$ terms algorithmically, for example it can be shown that

$$\gamma_0 = \gamma_1 = 1, \quad \gamma_2 = \frac{2m-n+3}{6}$$
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$$\gamma_0 = \gamma_1 = 1, \quad \gamma_2 = \frac{2m - n + 3}{6}$$

We know $\gamma_k$ has degree $k - 1$ in $(m, n)$, but an explicit formula is not currently known.
Terms needed for error < $1/1000$ for approximation of larger root of $f(x) = 1 - cx^2 + x^{13}$ for varying values of $c$:

<table>
<thead>
<tr>
<th>c</th>
<th>$x_{hi}$</th>
<th>$x_{sing}^+$</th>
</tr>
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<tbody>
<tr>
<td>1.5362173</td>
<td>$&gt;20000$</td>
<td>2</td>
</tr>
<tr>
<td>1.7</td>
<td>17</td>
<td>5</td>
</tr>
<tr>
<td>2.5</td>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>
Thank you!