

## SOLUTION OF LINEAR PROGRAMMING PROBLEMS

**THEOREM 1** If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set,  $S$ , associated with the problem. Furthermore, if the objective function  $P$  is optimized at two adjacent vertices of  $S$ , then it is optimized at every point on the line segment joining these two vertices, in which case there are infinitely many solutions to the problem.

**THEOREM 2** Suppose we are given a linear programming problem with a feasible set  $S$  and an objective function  $P = ax+by$ . Then,

If  $S$  is bounded then  $P$  has both a maximum and minimum value on  $S$

If  $S$  is unbounded and both  $a$  and  $b$  are nonnegative, then  $P$  has a minimum value on  $S$  provided that the constraints defining  $S$  include the inequalities  $x \geq 0$  and  $y \geq 0$ .

If  $S$  is the empty set, then the linear programming problem has no solution; that is,  $P$  has neither a maximum nor a minimum value.

### THE METHOD OF CORNERS

Graph the feasible set (region),  $S$ .

Find the EXACT coordinates of all vertices (corner points) of  $S$ .

Evaluate the objective function,  $P$ , at each vertex

The maximum (if it exists) is the largest value of  $P$  at a vertex. The minimum is the smallest value of  $P$  at a vertex. If the objective function is maximized (or minimized) at two vertices, it is minimized (or maximized) at every point connecting the two vertices.

Example: Find the maximum and minimum values of  $P=3x+2y$  subject to

$$x + 4y \leq 20$$

$$2x + 3y \leq 30$$

$$x \geq 0 \quad y \geq 0$$

1. Graph the feasible region.

Start with the line  $x+4y=20$ . Find the intercepts by letting  $x=0$  and  $y=0$  : (0,5) and (20,0).

Test the origin:  $(0) + 4(0) \leq 20$  . This is true, so we keep the half-plane containing the origin.

Graph the line  $2x+3y=30$ . Find the intercepts (0,10) and (15,0).

Test the origin and find it true. Shade the upper half plane.

Non-negativity keeps us in the first quadrant.

2. Find the corner points. We find (0,0) is one corner, (0,5) is the corner from the y-intercept of the first equation, The corner (15,0) is from the second equation. We can find the intersection of the two lines using intersect on the calculator or using rref: Value is (12, 2)

3. Evaluate the objective function at each vertex. Put the vertices into a table:

Vertex	$P=3x+2y$	
(0, 0)	0	min
(0, 5)	10	
(15, 0)	45	
(12, 2)	40	Max

4. The region is bounded, therefore a max and a min exist on  $S$ . The minimum is at the point (0,0) with a value of  $P=0$  . The maximum is at the point (15,0) and the value is  $P=45$  .

Example: A farmer wants to customize his fertilizer for his current crop. He can buy plant food mix A and plant food mix B. Each cubic yard of food A contains 20 pounds of phosphoric acid, 30 pounds of nitrogen and 5 pounds of potash. Each cubic yard of food B contains 10 pounds of phosphoric acid, 30 pounds of nitrogen and 10 pounds of potash. He requires a minimum of 460 pounds of phosphoric acid, 960 pounds of nitrogen and 220 pounds of potash. If food A costs \$30 per cubic yard and food B costs \$35 per cubic yard, how many cubic yards of each food should the farmer blend to meet the minimum chemical requirements at a minimal cost? What is this cost?

Before we can use the method of corners we must set up our system. Start by defining your variables properly and completely:

$x$  = no. of cubic yards of plant food mix A  
 $y$  = no. of cubic yards of plant food mix B

Next find the objective function. This is the quantity to be minimized or maximized:

Minimize  $C=30x+35y$

Finally, set up the constraints on the problem:

Subject to

$20x + 10y \geq 460$  phosphoric acid requirement

$30x + 30y \geq 960$  nitrogen requirement

$5x + 10y \geq 220$  potash requirement

$x \geq 0$   $y \geq 0$

Now use the method of corners:

1. Graph the feasible region. Start by putting in the three lines and finding their intercepts. Test the origin in each inequality and find that the origin is false, so we shade the lower half-plane of each. This feasible region is unbounded. That means that there is a minimum, but no maximum.

2. Find the corner points. We have 4 corners of the feasible region.

3. Evaluate the objective function at each vertex. Put the vertices into a table:

Vertex	$C=30x+35y$	
(0, 46)	1610	
(44, 0)	1320	
(20, 12)	1020	minimum
(14, 18)	1050	

4. The minimum occurs at (20,12). So the farmer should buy 20 cubic yards of plant food mix A and 12 cubic yards of plant food mix B at a cost of \$1020.

Example: Find the maximum value of  $f=10x+8y$  subject to

$$2x + 3y \leq 120$$

$$5x + 4y \leq 200$$

$$x \geq 0 \quad y \geq 0$$

1. Graph the feasible region.

$2x+3y=120$  Intercepts are (0,40) and (60,0). Origin is true so shade the upper half plane.

$5x+4y=200$  Intercepts are (0,50) and (40,0). Origin is true so shade the upper half plane.

2. Find the corner points. We find (0,0) is one corner, (0,40) is the corner from the y-intercept of the first equation. The corner (40,0) is from the second equation. We can find the intersection of the two lines using the intersection or rref:  $(120/7, 200/7)$ .

3. Evaluate the objective function at each vertex. Put the vertices into a table:

Vertex	$F=10x+8y$	
(0, 0)	0	
(0, 40)	320	
(40, 0)	400	Max
$(120/7, 200/7)$	400	Max

4. We find that the objective function is maximized at two points, therefore it is maximized along the line segment connecting these two points. The value of the objective function is  $400 = f = 10x + 8y$ . Write this in slope-intercept form,  $y = -1.25x + 50$  and  $120/7 \leq x \leq 40$ .

When you can't find the corners of the feasible region graphically (or don't want to!), we can use the simplex method to find the corners algebraically. The section we cover is for STANDARD MAXIMIZATION PROBLEMS. That is, the linear programming problem meets the following conditions:

The objective function is to be maximized.

All the variables are non-negative

Each constraint can be written so the expression involving the variables is less than or equal to a non-negative constant.

#### THE SIMPLEX METHOD:

1. Set up the initial tableau

Take the system of linear inequalities and add a slack variable to each inequality to make it an equation. Be sure to line up variables to the left of the '='s and constants to the right.

Rewrite the objective function in the form  $-c_1x_1 - c_2x_2 - \dots - c_nx_n + P = 0$ . Note the variables are all on the same side of the equation as the objective function and the objective function gets the positive sign.

Put this linear equations into an augmented matrix. This is the initial simplex tableau.

2. Look at the bottom row to the left of the vertical line. If all the entries are NON-NEGATIVE, simplex is done. go to step 4. If there are negative entries, the pivot element must be selected.

3. Select the pivot element

Find the most negative entry in the bottom row to the left of the vertical line. This is the pivot column. Mark it with an arrow.

find the quotients of the constant column value divided by the pivot column number,  $q = a/b$ .

Only positive numbers are allowed for b.

Choose the row with the smallest non-negative quotient. This is the pivot row.

Perform the pivot

Go to step 2

4. Solution - write out the equations row by row. Remember, if a column is messy it is a NB variable (parameter) and has a value of 0.

The simplex algorithm moves us from corner to corner in the feasible region (as long as the pivot element is chosen correctly). In this first example, I will show how the augmented matrix corresponds to a corner of the feasible region. In problems with more than two variables, you just must trust it to work.

Example: Maximize  $f=3x+2y$  subject to

$$x+y \leq 4$$

$$-2x+y \leq 2$$

$$x \geq 0 \quad y \geq 0$$

Introduce a slack variable into each inequality to make an equation. Rewrite the objective function into an equation:

$$x + y + u = 4$$

$$-2x + y + v = 2$$

$$-3x + 2y + f = 0$$

Set up the initial simplex tableau:

x	y	u	v	f	
1	1	1	0	0	4
-2	1	0	1	0	2
-3	-2	0	0	1	0

What are the values of our variables here? Remember, a column that is not a unit column (one 1 and the rest zero) is a non-basic variable and is equal to 0. We find the following:

$$x=0, \quad y=0, \quad u=4, \quad v=2 \quad \text{and} \quad f=0.$$

Look now at the feasible region. This point is at the origin. Can we increase  $f$ ? Yes, and changing  $x$  will increase it fastest. So we want to pivot on the  $x$  column. Look at the quotients. Only one is possible. So we will pivot on the first row. Use the calculator program SMPLX. What we find is

x	y	u	v	f	
1	1	1	0	0	4
0	3	2	1	0	10
0	1	3	0	1	12

What are the values of the variables now?  $x=4$ ,  $y=0$ ,  $u=0$ ,  $v=10$ , and  $f=12$ .

What has happened on the graph? We have moved to the corner point  $(4,0)$ . We couldn't have moved to the other corner point on the  $x$ -axis as it is out of the feasible region (that's what the negative number would cause). By choosing the smallest non-negative quotient we move to a corner of the feasible region, not just any intersection point.

Are we done? There are no more negative numbers in the bottom row and so we have found the maximum value of  $f$ .

Example: In the following matrices, find the pivot element if simplex is not done. Find the values of all the variables if simplex is done.

$$(a) \begin{array}{cccc|c} x & y & u & v & P \\ 1 & 1 & 1 & 0 & 0 & 10 \\ 2 & 1 & 0 & 1 & 0 & 20 \\ -3 & 2 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on row1 column 1 or row 2 column 1 as in a tie you can choose either

$$(b) \begin{array}{cccc|c} x & y & u & v & P \\ -1 & 1 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 & 20 \\ -3 & 2 & 0 & 0 & 1 & 0 \end{array}$$

NO SOLUTION (nothing to pivot on and there is a negative number in the bottom row)

$$(c) \begin{array}{cccc|c} x & y & u & v & P \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 2 \\ -3 & -2 & 0 & 0 & 1 & 0 \end{array}$$

pivot on row 1 column 1

$$(d) \begin{array}{cccc|c} x & y & u & v & P \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 2 & 0 & 0 & 1 & 0 & 2 \\ 3 & -2 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on row 1 column 2

$$(e) \begin{array}{cccc|c} x & y & u & v & P \\ 0 & 0 & 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 10 & 1 & 10 \end{array}$$

Example: A company makes three models of desks, an executive model, an office model and a student model. Each desk spends time in the cabinet shop, the finishing shop and the crating shop as shown in the table:

Type of desk	Cabinet shop	Finishing shop	Crating shop	Profit
Executive	2	1	1	150
Office	1	2	1	125
Student	1	1	.5	50
Available hours	16	16	10	

How many of each type of model should be made to maximize profits?

Answer: Start by defining your variables:

$x$  = number of executive desks made

$y$  = number of office desks made

$z$  = number of student desks made

Maximize  $P=150x+125y+50z$  .subject to

$2x + y + z \leq 16$  cabinet hours

$x + 2y + z \leq 16$  finishing hours

$x + y + .5z \leq 10$  crating hours

$x \geq 0$  ,  $y \geq 0$  ,  $z \geq 0$  .

We will give each of the inequalities a slack variable. Write the initial tableau,

$x$	$y$	$z$	$u$	$v$	$w$	$P$	
2	1	1	1	0	0	0	16
1	2	1	0	1	0	0	16
1	1	.5	0	0	1	0	10
-150	-125	-50	0	0	0	1	0

Use the simplex algorithm to find the pivot elements. Choose row 1, column 1 for our first pivot element. The second pivot will be on row 3 column 2. our final tableau looks like

$x$	$y$	$z$	$u$	$v$	$w$	$P$	
1	0	.5	1	0	-1	0	6
0	0	.5	1	1	-3	0	2
0	1	0	-1	0	2	0	4
0	0	25	25	0	100	1	1400

Set the NB variables to zero,  $z=0$  ,  $u=0$  ,  $w=0$  .

Now read across each row to find  $x=6$  ,  $y=4$  ,  $v=2$  and  $P=1400$ .

This is a word problem and needs to be explained:

Make 6 executive desks, 4 office desks and no student desks. All of the hours in the cabinet shop and crating shop will be used. There will be 2 hours of finishing shop unused. The profit will be \$1400.