Math 131 Week-in-Review #6 (Sections 3.2-3.4)

1. Find the derivative of each of the following functions. Do not simplify your answers.

a) \( f(x) = \left(2x^2 - 4x + 7\right)\left(\frac{4}{x^9} + 3^x - \sec x\right) \Rightarrow F S' + S F' \)

\[
f'(x) = \left(2x^2 - 4x + 7\right) \left[-36x^{-10} + (\ln 3)3^x - \sec x \tan x\right] + \\
\left(4x^{-9} + 3^x - \sec x \sec x\right) \left[4x - 4\right]
\]
b) \[ f(x) = \frac{(x^3 - 7x + \pi^2)e^x}{3\sqrt[3]{x^3} - x^4 + 2x - 3} \]

\[ f'(x) = \left[ (3x^{3/5} - x^4 + 2x - 3) \left( (x^3 - 7x + \pi^2)[e^x] + (e^x)[3x^2 - 7] \right) \right] \]

\[ \frac{\frac{\partial T'}{\partial B'} - \frac{\partial T}{\partial B}}{B^3} \]
c) $f(x) = \frac{rx^2 + n}{m x + p} \quad \text{①}\quad \text{②}$

$r, n, m, p$ are constants!

$$f'(x) = \left[ \frac{(mx^{-1} + p) [2rx]}{(mx^{-1} + p)^2} - \frac{(rx^2 + n) [-mx^{-2}]}{(mx^{-1} + p)^2} \right]$$
d) \( f'(\theta) = \frac{3e^\theta + \csc(\theta)}{1 + \sin(\theta)} + 98^x \cos(\theta) \cot(\theta) \)

\[
f'(\theta) = \left[ \frac{(1 + \sin(\theta) [3e^\theta - \csc \theta \cot \theta] - (3e^\theta + \csc \theta) \cos \theta}{(1 + \sin(\theta))^2} \right]
\]

\[
+ \left( 98^x \cot \theta \left[ -\csc^2 \theta \right] + (\cot \theta) [98^x [- \sin \theta] + \cos \theta \ln 98 (98^x)] \right)
\]
2. Joe, a farmer in West Texas, is converting all of his crops to cotton. It is not feasible for him to convert all of his crops to cotton at once. He is currently farming 400 acres of cotton this year and plans to increase that number by 70 acres per year. As he becomes more experienced growing cotton, his output increases. He currently yields an average of 600 pounds of cotton per acre, but this average yield is increasing by 50 pounds per acre per year. How rapidly is the total number of pounds of cotton increasing per year?

\[
\begin{align*}
\text{Affected by} & \quad \{ \text{\# of acres, } A(t) \} \\
\text{\# of pounds of cotton per acre, } P(t) \\
\text{Where } t \text{ is \# of yrs. from now.} \\
\end{align*}
\]

\[
C(t) = A(t)P(t)
\]

\[
C'(0) = A(0)P'(0) + P(0)A'(0)
\]

\[
= 400 \cdot 50 + 600 \cdot 70 = 62,000 \text{ pounds yr.}
\]

\[
\text{\# of acres now, rate of increase \# of acres, rate of increase \# of pounds/acre, rate of increase \# of acres.}
\]

\[
\text{\# of pounds of cotton is increasing by 62,000 pounds/yr.}
\]
3. The position of a particle is given by \( s = t^3 - 10.5t^2 + 30t, \ t \geq 0 \), where \( t \) is time in seconds and \( s \) is measured in meters.

a) Find the velocity and acceleration after 4 seconds.

\[
\begin{align*}
V(4) &= s'(4) = 3t^2 - 21t + 30 \\
S'(4) &= \left[-6 \text{ m/s}\right] \text{ (velocity)} \\
&\quad \text{* neg \Rightarrow moving backward (neg. direction)} \\
&\quad \text{* gives no change in position (decreasing)} \\
&\quad \text{* speed = 6 m/s}.
\end{align*}
\]

\[
\begin{align*}
a(4) &= V'(4) = S''(4) \\
S''(4) &= 6t - 21 \\
S''(4) &= \left\{ \frac{3 \text{ m/s}^3}{3 \text{ m/s}^3} \right\} \text{ (accel)} \\
&\quad \text{* gives change in velocity (increasing)} \\
&\quad \text{* pos. \Rightarrow pushing object in opposite direction if it is moving.} \\
&\quad \text{* \( V(t) < 0 \) \Rightarrow moving backward \& slowing down.}
\end{align*}
\]

b) When is the particle at rest?

\[
\begin{align*}
V(t) &= 0 \\
3(t^2 - 7t + 10) &= 0 \\
3(t - 2)(t - 5) &= 0 \\
&\Rightarrow \left\{ \begin{array}{c} t = 2 \text{ sec.} \\
\text{or} \\
\text{t} = 5 \text{ sec.} \end{array} \right.
\end{align*}
\]
c) When is the particle moving forward (that is, in the positive direction)?

\[ v(t) > 0 \Rightarrow 3(t-2)(t-5) > 0 \]

\[ \begin{align*}
\text{moving forward:} & \quad 0 \leq t < 2 \\
& \quad t > 5 \\
\text{backward:} & \quad 2 < t < 5 
\end{align*} \]

d) Find the total distance traveled by the particle during the first 4 seconds.

Cannot see \( s(t) \)!

\[ \begin{align*}
\text{ } & \quad t=5 \\
\text{ } & \quad s=12.5 \\
\text{ } & \quad t=4 \\
\text{ } & \quad s=16 \\
\text{ } & \quad t=2 \\
\text{ } & \quad s=26 \\
\end{align*} \]

\[ \begin{align*}
|s(2) - s(0)| &= |126 - 0| = 126 \text{ m} \\
|s(4) - s(2)| &= |16 - 26| = 10 \text{ m} \\
2(26) + 10 &= \boxed{36 \text{ m}} 
\end{align*} \]
e) Determine when the particle is speeding up and/or slowing down. **Hint:** Use the graphs of the position, velocity, and acceleration functions below. Be sure to **explain** your reasoning.

- **Speeding up:**
  - $v(t) \cdot a(t)$ have same sign.
  - **(pushed in same direction it is moving.)**
  - *Speeding up + moving forward* $v(t) > 0 + a(t) > 0$
    - $t > 5 \text{ sec}$
  - *Speeding up + moving backward* $v(t) < 0$ and $a(t) < 0$
    - $2 < t < 3.5 \text{ sec}$

- **Slowing down**
  - $v(t) + a(t)$ have opp. signs.
  - **(pushed in opp. direction it is moving)**
  - *Slowing down + moving forward* $v(t) > 0 + a(t) < 0$
    - $0 \leq t < 2 \text{ sec}$
  - *Slowing down + moving backward* $v(t) < 0 + a(t) > 0$
    - $3.5 < t \leq 5 \text{ sec}$
4. Use calculus to determine on what intervals $f(x) = x - 3 \sin x$ is increasing/decreasing and concave upward/downward on $0 \leq x \leq 2\pi$. **Hint:** Graph the appropriate derivatives and approximate the intervals.

**Increasing/Decreasing**
\[
f'(x) = 1 - 3\cos x
\]
Increasing on $(1.2310, 5.0522)$
Decreasing on $(0, 1.2310)$ and $(5.0522, 2\pi)$

**Concavity**
\[
f''(x) = 3\sin x
\]
Concave up on $(0, \pi)$
Concave down on $(\pi, 2\pi)$
5. Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where $F$ and $G$ are the functions whose graphs are shown below. (Source: #36, pg. 199, Stewart)

a) Find $P'(2)$.

$$P'(2) = F(2)G'(2) + G(2)F'(2)$$

$$= 3 \cdot \frac{1}{2} + 2 \cdot 0 = \frac{3}{2}$$

**Aside:** $G'(2) = \frac{\text{rise}}{\text{run}} = \frac{2}{4} = \frac{1}{2}$

b) Find $Q'(7)$.

$$Q'(7) = \frac{G(7)F'(7) - F(7)G'(7)}{G(7)^2}$$

$$= \frac{1 \cdot (\frac{4}{7}) - 5 \cdot (-\frac{3}{5})}{(G(7))^2} = \frac{43}{10}$$

**Aside:** $F'(7) = \frac{4}{7}$, $G'(7) = -\frac{3}{5}$
6. Find equations of the tangent lines to the curve \( y = \frac{x - 1}{x + 1} \) that are parallel to the line \( x - 2y = 2 \).

\[
M_1 = y' = \frac{(x+1)x - (x-1)x}{(x+1)^2} = \frac{2}{(x+1)^2}
\]

\[
M_2 = -\frac{\alpha}{\beta} = -\frac{1}{2} = \frac{1}{2} \quad \text{(standard form)}
\]

\[
M_1 = M_2 \quad \Rightarrow \quad \frac{2}{(x+1)^2} = \frac{1}{2} \quad \Rightarrow \quad 4 = (x+1)^2 \quad \Rightarrow
\]

\[
\pm 2 = x + 1 \quad \Rightarrow \quad x = 1 \quad \text{and} \quad x = -3
\]

\[
\chi = 1 \quad \Rightarrow \quad y = \frac{1 - 1}{1 + 1} = 0 \quad \Rightarrow \quad (1, 0) \quad m = \frac{1}{2}
\]

\[
y - 0 = \frac{1}{2} (x - 1)
\]

\[
y = \frac{1}{2} x - \frac{1}{2}
\]

\[
\chi = -3 \quad \Rightarrow \quad y = \frac{-3 - 1}{-3 + 1} = \frac{2}{2} = 1 \quad \Rightarrow \quad (-3, 2) \quad m = \frac{1}{2}
\]

\[
y - 2 = \frac{1}{2} (x + 3)
\]

\[
y = \frac{1}{2} x + \frac{7}{2}
\]
7. Use calculus to determine on what intervals the function \( f(x) = x^2 e^x \) is increasing/decreasing and concave upward/downward.

0. Min. \( \Rightarrow f'(x) > 0 \Rightarrow f'(x) = x^2 e^x + e^x (2x) \Rightarrow e^x (x^2 + 2x) > 0 \)

\( e^x \) is always > 0 \( \Rightarrow \) need \( x^2 + 2x > 0 \)

\( \Rightarrow x(x+2) > 0 \Rightarrow \begin{cases} x < -2 & \text{or} & x > 0 \end{cases} \)

\[ \begin{array}{c}
\text{Min. on} \ (-\infty, -2) \text{ and } (0, \infty) \\
\text{Dec. on} \ (-2, 0)
\end{array} \]

1. Conc. up \( \Rightarrow f''(x) > 0 \Rightarrow f''(x) = e^x (2x + 2) + (x^2 + 2x) e^x \)

\( f''(x) = e^x (x^2 + 4x + 2) > 0 \)

\( x^2 + 4x + 2 > 0 \).

\[ x = \frac{-4 \pm \sqrt{16 - 8}}{2} = \frac{-4 \pm \sqrt{8}}{2} \Rightarrow x = -2 \pm \sqrt{2} \]

\[ \begin{array}{c}
\begin{cases}
-2 - \sqrt{2} \\
-2 + \sqrt{2}
\end{cases} \quad \text{and} \quad \begin{cases}
-3.4 \\
-0.6
\end{cases}
\end{array} \]

Concave up on \( (-\infty, -2 - \sqrt{2}) \) and \( (-2 + \sqrt{2}, \infty) \)

Concave down on \( (-2 - \sqrt{2}, -2 + \sqrt{2}) \)
8. Find the equation of the line tangent to the curve $y = \tan(x)$ at $x = \pi/3$. **Hint:**

$\sec x = \frac{1}{\cos x}, \tan x = \frac{\sin x}{\cos x}, \sin(\pi/3) = \frac{\sqrt{3}}{2},$ and $\cos(\pi/3) = \frac{1}{2}$.

$m = y' = \sec^2 x \Rightarrow y'(\pi/3) = \sec^2(\pi/3)$

$= \left(\frac{1}{\cos(\pi/3)}\right)^2 = \left(\frac{1}{\frac{1}{2}}\right)^2 = 4 = m$

$y$ coord. $\Rightarrow y = \tan(\pi/3) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2}$

$= \sqrt{3} \cdot 2 = 2\sqrt{3} = y_0$

$y - \sqrt{3} = 4(x - \pi/3)$ or $y = 4x - \frac{4\pi}{3} + \sqrt{3}$
9. A penny is dropped into the bottom of a fountain, creating a circular ripple that travels outward at a speed of 30 cm/s. Find the rate at which the area within the circle is increasing after 1 second, 3 seconds, and 5 seconds. What can you conclude?

After $t$ seconds, the radius is $r = 30t$.

So area is $A(t) = \pi (30t)^2 = 900 \pi t^2$ cm$^2$.

$A'(t) = 1800 \pi t$ cm$^2$/s

$A'(1) = \frac{1800 \pi \text{ cm}^2}{5}$ $A'(3) = \frac{5400 \pi \text{ cm}^2}{5}$

$A'(5) = \frac{9000 \pi \text{ cm}^2}{5}$

As time goes by, area grows at an increasing rate. The rate of change is linear with respect to time.
10. Fill in the blanks below.

- Given a charge function $Q(t)$ that gives the electric charge at time $t$, the \textit{current} \( I \) is the rate of change of the \textit{charge} \( Q \) with respect to \textit{time} and is found by calculating \( \frac{dQ}{dt} \).

- Given a mass function $f(x)$ that gives the mass of a rod at length $x$, the rod’s \textit{linear density} \( \lambda \) is the rate of change of the \textit{mass} \( f \) with respect to \textit{length} and is found by calculating \( \frac{df}{dx} \).

- Given a volume function $V(P)$ that gives the volume of a substance with pressure $P$, the substance’s \textit{compressibility} \( \beta \) is the rate of change of the \textit{volume} \( V \) with respect to \textit{pressure}, per unit volume, and is found by calculating \( \beta = -\frac{1}{V} \frac{dV}{dP} \).

- Given a population function $n(t)$ that gives the population at time $t$, the population’s \textit{growth rate} is the rate of change of the \textit{population} with respect to \textit{time} and is found by calculating \( \frac{dn}{dt} \).
• Given the law of laminar flow \( v = \frac{P}{4\eta l} (R^2 - r^2) \), which gives the velocity of the blood through a vessel or artery based on its distance from the central axis, the blood’s velocity gradient is the rate of change of the velocity with respect to \( r \) (i.e., distance from central axis), and is found by calculating \( \frac{dv}{dr} \).

• Given a cost function \( C(x) \) that gives the cost of producing \( x \) units, the marginal cost is the rate of change of the total cost with respect to \( x \) (number of units produced), and is found by calculating \( \frac{dC}{dx} \).