Homework 12

Math 300 (section 901), Fall 2021

This homework is due on Wed., Nov. 17. (Turn in your answers to questions 1–5.)
You may cite results from class, as appropriate.

0. *(This problem is not to be turned in.)*
   (a) Explain what is wrong with the following: *Consider a function* $f : \mathbb{Z} \to 9$.
   (b) Give an example of a function $f : \mathbb{Z} \to \mathbb{R}$.
   (c) Give an example of a function $f : \mathbb{R} \to \mathbb{Q}$.
   (d) What is the difference between $f(x)$, where x is an element, and $f(X)$, where X is a set?

1. (No proofs necessary for this problem)
   (a) List *all* functions $f : \mathbb{Z} \to \{8\}$ (functions with domain $\mathbb{Z}$ and codomain $\{8\}$).
   (b) List *all* **one-to-one** (injective) functions $f : \{0, 1\} \to \{2, 3, 4\}$.
   (c) List *all* **onto** (surjective) functions $f : \{0, 1\} \to \{2, 3\}$.

2. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n$ if n is even and $f(n) = n - 3$ if n is odd.
   (a) *Prove or disprove*: f is one-to-one.
   (b) *Prove or disprove*: f is onto.

3. Let $f : A \to C$ and $g : B \to D$ be functions. Consider the following function[1]:

$$\begin{aligned} h \ : \ & A \times B \to C \times D \\ & (a, b) \mapsto (f(a), g(b)) \ . \end{aligned}$$

   (a) *Prove or disprove*: If f and g are one-to-one, then so is h.
   (b) *Prove or disprove*: If f and g are onto, then so is h.

4. Let A be a nonempty set. Assume $b \notin A$. Consider the following function:

$$\begin{aligned} h \ : \ & \mathcal{P}(A) \to \mathcal{P}(A \cup \{b\}) \\ & S \mapsto S \cup \{b\} \ . \end{aligned}$$

   (a) *Prove or disprove*: h is one-to-one.
   (b) *Prove or disprove*: h is onto.

5. Let $f : A \to B$ be a function, and let $C \subseteq A$ and $D \subseteq B$. *Prove or disprove the following:*
   (a) $f(f^{-1}(D)) \subseteq D$
   (b) $f(f^{-1}(D)) \supseteq D$
   (c) $f^{-1}(f(C)) \subseteq C$
   (d) $f^{-1}(f(C)) \supseteq C$

[1] In #3 and #4, we use the notation $x \mapsto y$ (for a function h), which means $h(x) = y$.