Homework 5

Math 300 (section 901), Fall 2021

This homework is due on Wed., Sept. 29. (Turn in your answers to questions 1–10.)
You may cite results from class, as appropriate.

0. *(This problem is NOT to be turned in.)*
   (a) Read Sections 3.3–3.5 and 4.1–4.2.
   (b) Section 3.2 #3.8
   (c) Section 3.3 #3.18
   (d) Section 3.4 #3.26, 3.30
   (e) Section 3.5 #3.42
   (f) Section 4.1 #4.1, 4.5
   (g) Section 4.2 #4.14, 4.15
   (h) Prove that an integer n is even if and only if $-n$ is even.
   (i) Conclude (explain why you can!) that an integer n is odd if and only if $-n$ is odd.
   (j) Prove that an integer n is even if and only if its last digit (the ones digit) is 0, 2, 4, 6, or 8. (*Hint*: For $n > 0$, consider the remainder after dividing by 10; for $n < 0$, use a previous problem.)
   (k) Conclude (explain why you can!) that an integer n is odd if and only if the last digit is 1, 3, 5, 7, or 9.

1. Rewrite the following using "for all" (or "for every")[1] and/or "there exist(s)".
   (a) Every even integer can be expressed as the sum of two odd integers.
   (b) The product of any three odd integers is odd.
   (c) The square of at least one real number is 0.
   (d) There is an integer whose square is negative.
   (e) Every real number can be written as the difference of an integer and a real number in the interval $[0, 1)$.

2. Negate your answers to #1.

3. Complete the following claim, and give a proof:
   Let n be an integer. Then $(n+1)(n-1)+3$ is even, if and only if n is ____________.

[1] Your answers can be sentences in English (e.g., "For every even integer n, there exist..."), or you can use symbols (e.g. "$\forall$...") if you prefer.

4. Complete the following claim, and give a proof: *Let a and b be integers. If $5a - 2b$ and $4a + 3b$ are both even, then a and b are both* ____________.

5. A student turns in the following proof: *Assume that p, q, and r are even integers. Then, by definition, there exists an integer k such that $p = 2k$ and $q = 2k$ and $r = 2k$. Therefore, $pqr = (2k)(2k)(2k) = 8k^3 = 2(k^3)$, by associativity and commutativity of integers. By closure, $4k^3$ is an integer and hence pqr is even (by definition).*

   (a) What statement (your best guess) was this student trying to prove?

   (b) Critique the proof.

6. Critique the following "proof" that every even integer is also odd: *Assume that x is even. Then, by definition, $x = 2k$ for some integer k. So, $x = 2k = 2(k - 1/2) + 1$. Hence, by definition, x is odd.*

7. (a) For which integers $n \geq 2$ is the following congruence true: $10 \equiv 1 (\text{mod } n)$? Explain your answer.

   (b) (Bonus problem – OPTIONAL) Use your answer to (a) to explain why you can check whether a positive integer is divisible by 3 or 9 by seeing whether the sum of its digits is divisible by, respectively, 3 or 9.

8. Prove the following:

   (a) *For integers m and n, if $m \equiv n (\text{mod } 10)$, then $m \equiv n (\text{mod } 5)$.*

   (b) *If m is an integer, then $(m^3 - m) \equiv 0 (\text{mod } 3)$.*

9. Suggest two problems for the first midterm exam (which is on Friday, October 8):

   - one from the Chapter 3 Supplementary Exercises, and
   - another one on any topic in Chapter 3 (please invent a problem, rather than taking one directly from the textbook).

10. (a) Section 3.2 #3.10

    (b) Section 3.5 #3.44

11. (Bonus problem – OPTIONAL) Rescale a right triangle with edge lengths 3, 4, 5 to find a point on the unit circle with nonzero, rational-number coordinates. (This answers a question from class, a from a few weeks ago.)