Homework 9

Math 300 (section 901), Fall 2021

This homework is due on Wed., Oct. 27[1]. (Turn in your answers to questions 1–6.)
You may cite results from class, as appropriate.

0. *(This problem is NOT to be turned in.)*
   (a) Read Sections 6.2–6.3
   (b) Are the following statements logically equivalent? (Explain your answer.)
      (i) When I drive, I don't text.
      (ii) I never drive and text.
   (c) Section 6.1 #6.5, 6.9, 6.11, 6.16
   (d) Section 6.2 #6.19, 6.29

1. Prove or disprove the following claims:
   (a) Every odd integer can be expressed as the product of two odd integers.
   (b) Every even integer can be expressed as the product of two even integers.
   (c) For real numbers x and y, if $xy \neq 0$, then $x \neq 0$.
   (d) Let n be an integer. If $2|(n^2 - 5)$, then $4|(n^2 - 5)$.
   (e) Let n be an integer. If $2|(n^2 - 5)$, then $8|(n^2 - 5)$.
   (f) Let n be an integer with $n \geq 2$. For every integer x, the following is true: x is odd if and only if x^n is odd.
   (g) For every nonnegative integer n, the following inequality holds: $3^n > n^2$.

2. Consider the statement, *For every nonnegative integer n, if A is a finite set of cardinality n, then the number of subsets of A is 2^n.*
   (a) State the **base case** for a proof (of the statement) by induction (on n).
   (b) Prove the base case.
   (c) State the **inductive hypothesis.**
   (d) State the *goal* of the **inductive step**.
   (e) To complete the inductive step, let A be a set of cardinality $k+1$, which we write as $A = \{x_1, x_2, \ldots, x_{k+1}\}$. Consider the set $B = \{x_1, x_2, \ldots, x_k\}$ (which is a subset of A). How are the subsets of A that do NOT contain the element x_{k+1} related to the subsets of B? Explain.
   (f) How are the subsets of A that contain x_{k+1} related to the subsets of B? Explain.
   (g) Use your answers to (e) and (f) – plus the inductive hypothesis – to count the total number of subsets of A.
   (h) Do your above answers prove that $|\mathcal{P}(A)| = 2^{|A|}$ for every finite set A? (Compare with Theorem 6.16 in your textbook.) Explain.

[1] As a reminder, your writing assignment (partial drafts of your final report) are also due on this day.

3. Consider a sequence $\{a_n\}$ defined recursively as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_k &= a_{k-1} + 2a_{k-2} \quad \text{for } n \geq 3 \ . \end{aligned}$$

   (a) Compute a_3, a_4, a_5, a_6. (Show your work.)

   (b) Conjecture a formula for a_n.

   (c) Prove your conjecture.

4. Assess (or give your opinion on) the following advice: *When proving that an equation is true (for instance, $1 \cdot 2 \cdot 3 \cdots n = n(n+1)/2$), start with one side of the equation and prove (usually through a sequence of equalities) that it equals the other side.*

5. Section 6.1 #6.8, 6.10, 6.12 (and explain your answer to 6.12(b))

6. Section 6.2 #6.30