Homework 13

Math 653, Fall 2019

This homework is due on Thursday, November 21.

1. Read Hungerford, sections 3.2 and 3.3
   (a) Theorem 2.18 (page 128) asserts that maximal ideals always exist. Why does the proof require Zorn's Lemma?
   (b) What is the product in the category of rings? (No proof necessary.)
   (c) How does the Chinese Remainder Theorem (Theorem 2.25) generalize the subsequent Corollary 2.26?
   (d) Section 3.2 #2, 13, 21
   (e) Section 3.3 #3, 4, 6, 7
   (f) *(These problems are not to be turned in.)* Section 3.2 #3–5, 10, 17
   (g) *(These problems are not to be turned in.)* Section 3.3 #1, 2, 13
   (h) *(This problem not to be turned in.)* Prove the Second and Third Isomorphism Theorems for rings (Theorem 2.12 on page 126).
   (i) *(This problem not to be turned in.)* Is there a left-ideal version of the correspondence theorem (Theorem 2.13 on page 126)?
   (j) *(This problem not to be turned in.)* Is 5 a unit in $\mathbb{Z}[x]$? What about in $\mathbb{Q}[x]$? Is $x + 1$ a unit in $\mathbb{Z}[x]$? What about in $\mathbb{Q}[x]$? Do the constant polynomials form a subring and/or an ideal of $\mathbb{Z}[x]$?
   (k) *(This problem not to be turned in.)* Prove or disprove: *Let I be an ideal of a ring R. If every ideal of R/I is a principal ideal, then every ideal of R that contains I is a principal ideal.*

2. Prove or disprove the following:
   Claim: Let R be a ring. If there exists a nonzero ring homomorphism $\phi : \mathbb{C} \to R$, then R contains a subring isomorphic to $\mathbb{C}$.

3. Let P be a prime ideal in a ring R, and assume that P contains the intersection of two ideals I and J. Prove that P contains I or P contains J.

4. List all ring homomorphisms $\mathbb{Z}_{10} \to \mathbb{Z}_8$. Give a proof.

5. Prove or disprove the following claims:
   (a) **Claim:** Let $\phi : R \to R'$ be a ring homomorphism. If R has a multiplicative identity 1, then $\phi(1)$ is the multiplicative identity of R'.
   (b) **Claim:** Let N and N' be ideals of a ring R. If $R/N \cong R/N'$, then $N = N'$.

6. Determine whether each of the following ideals (given by their generating sets) is prime/maximal/principal in $\mathbb{Z}[x]$. Prove your answers.
   (a) $(4,\ x)$
   (b) $(x^2 - 1)$
   (c) (5)
   (d) $(2x^3 + x^2 - 14x - 7,\ x^3 + x^2 - 7x - 7)$
7. (a) Determine whether the following ring is a field, and give a proof:
   $$\mathbb{Q}[x, y]/\langle y - 1,\ x + y + 2\rangle.$$
   (Here, $\mathbb{Q}[x, y]$ is the ring of polynomials in two variables, x, and y; one such polynomial is $f(x) = x^3y - 1/3$.)
   (b) Is the ideal $\langle y - 1,\ x + y + 2\rangle$ a maximal ideal of $\mathbb{Q}[x, y]$? Is it a prime ideal? Explain your answers.
8. For each problem that you missed on the exam, re-do the problem. (If your answer on the exam was partially correct, you may write "On the exam, I proved . . . ; what remains to prove is . . . ".)
9. Write one paragraph reflecting on the exam. *How did you prepare for the exam? Did you study alone or with other students? What surprised you about the exam? How do you feel about your performance on the exam? Which types of errors (if any) did you make (not understanding the problem, not being able to formulate a solution, etc.)? Are there any concepts you want to work on before the next exam? Any other reflections or comments?*