Homework 5

Math 653, Fall 2019

This homework is due on Thursday, September 26.

1. Read Hungerford, sections 1.7–1.8.
   (a) Section 1.7 #2, 5
   (b) Section 1.8 #1, 5, 6
   (c) *(These problems are not to be turned in.)* Section 1.7 #1, 3, 6
   (d) *(These problems are not to be turned in.)* Section 1.8 #9, 11
2. *Prove or disprove*: Let G be a group. If $a \in G$ and $b \in G$ both have order 2, then ab has order 1, 2, or 4.
3. (a) Is S_n generated by every set consisting of all 3-cycles and a single transposition? Explain.
   (b) Is S_n generated by all 4-cycles? Prove your answer.
4. The Third Isomorphism Theorem gives an isomorphism between $(\mathbb{Z}/36\mathbb{Z})/(6\mathbb{Z}/36\mathbb{Z})$ and $\mathbb{Z}/6\mathbb{Z}$.
   (a) Give a formula for this isomorphism.
   (b) Are there other isormorphisms between these two groups? If so, list them (and explain why your list is complete). If not, explain why not.
5. On Homework 2, you showed that $H_n := \{\sigma \in S_n \mid \sigma(n) = n\}$ is a subgroup of S_n that is isomorphic to S_{n-1}.
   (a) Conjugate H_n by the transposition $(1, n)$. Which subgroup of S_n do you get? (Prove your answer.)
   (b) Is H_n normal in S_n? Explain.
6. Determine (and then prove) for which positive integers n the following claim is true: *Every index-n subgroup is normal.*.
7. *For each of the following, either give an example of such a group or prove that no such group exists.*
   (a) An infinite group in which every non-identity element has order 2.
   (b) An infinite group in which every non-identity element has order 2 or 4.
   (c) An infinite group in which every non-identity element has order 2 or 6.
   (d) A group in which every element has finite order, and for every positive integer n there is an element of order n.

(e) A non-trivial group G such that G is isomorphic to $G \times G$.

(f) A group G such that every finite group is isomorphic to some subgroup of G.

(g) For a positive integer m, a group G_m with exactly m normal subgroups.

8. Are there groups G_1, G_2, H_1, and H_2, such that $G_1 \times G_2$ is isomorphic to $H_1 \times H_2$, but no G_i is isomorphic to any H_j? Prove your answer.

9. Suggest a problem for the first exam (which is on Thursday, October 3).