Math 152 Week-in-Review

Final Exam Review ~~Monday, May 1, 1-3pm HELD 100~~

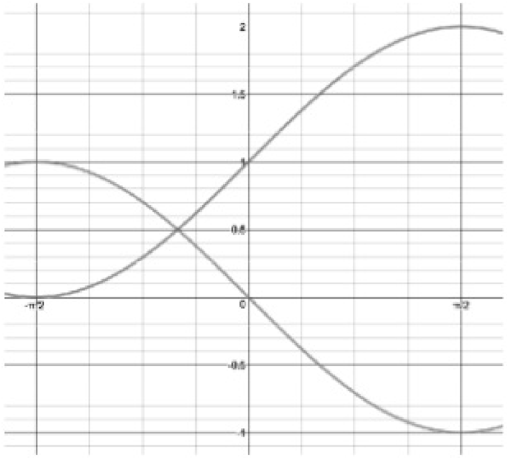
Today, This room, 7:00 – until I get tired

We don't have time to work all the questions,
so I will let you tell me which ones to work.

1. Evaluate $\int_0^{(\pi^2/9)} \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$.

2. Find the area of the region bounded by $y = x^2 - 2x$ and $y = 2x - 3$.

3. Write an integral for the area of the region bounded by $y = -\sin x$ and $y = 1 + \sin x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$. Do not evaluate.



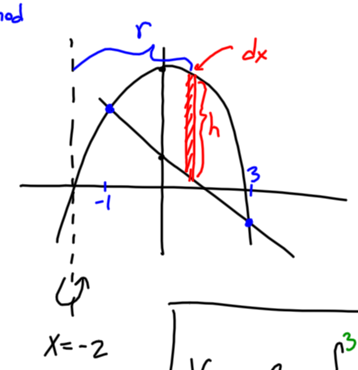
4. Consider the region bounded by $y = x^3$, $x = 0$, $y = 8$ and $y = 1$. Find the volume if this region is revolved around the y-axis.

5. Consider the region bounded by $y = 4 - x^2$ and $y = -2x + 1$. Set up an integral, but do not evaluate, for the volume if the region is rotated around the following lines.

Shell Method

(a) $x = -2$

(b) $y = 4$



$4 - x^2 = -2x + 1$

$0 = x^2 - 2x - 3$

$(x-3)(x+1) = 0$

$x = -1, \quad x = 3$

$r = x - (-2)$

$r = x + 2$

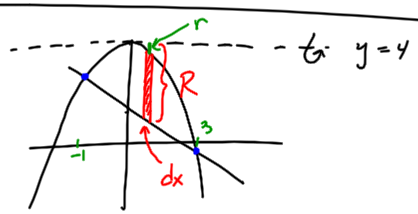
$h = (4 - x^2) - (-2x + 1)$

$h = 3 - x^2 + 2x$

$$\boxed{V = 2\pi \int_{-1}^{3} (x+2)(3 - x^2 + 2x)\, dx}$$

$R = 4 - (-2x + 1)$

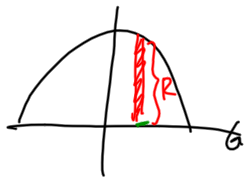
$R = 3 + 2x$



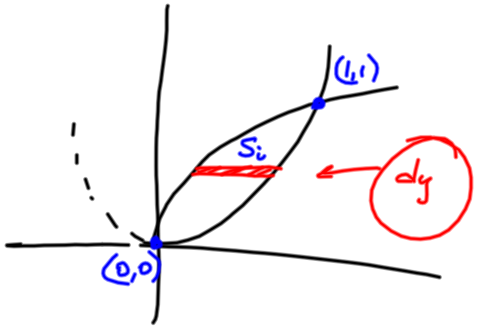
$r = 4 - (4 - x^2)$

$r = x^2$

$$\boxed{V = \pi \int_{-1}^{3} (3 + 2x)^2 - (x^2)^2 \, dx}$$



6. The solid S has a base bounded by the functions $y = x^2$ and $y = \sqrt{x}$. Cross-sections perpendicular to the y-axis are squares. Set up an integral for the volume of S, but do not evaluate.



$A_i = (S_i)^2$

$V = \int (S_i)^2 \, dy = \boxed{\int_0^1 \left[\sqrt{y} - y^2\right]^2 dy}$

$S_i = \sqrt{y} - y^2$

$y = x^2$

$x = \sqrt{y}$

$y = \sqrt{x}$

$x = y^2$

7. A spring has a natural length of 3 m. The force required to keep the spring stretched to a length of 8 m is 7 N. Find the work required to stretch the spring from a length of 4 m to a length of 6 m.

$\Delta x = 1$ $\Delta x = 3$

(a) Find k

Spring Force $f(x) = k \cdot x$ The distance from natural length

Stretch 5 m (From 3 to 8) $7 = k \cdot 5$ $k = \frac{7}{5}$

(b) answer question $W = \int f(x)\,dx = \int_1^3 \frac{7}{5}x\,dx$

$$\frac{7}{5}\frac{x^2}{2}\Big|_1^3 = \frac{7}{10}(8) = \boxed{5.6}$$

8. A rope that weighs $\frac{1}{6}$ pounds per foot is used to pull a bucket full of water to the top of a 60 ft well. The bucket weighs 15 pounds when full of water. How much work is done?

$$W = \int f(x)\,dx = \boxed{\int_0^{60} 25 - \frac{1}{6}x \quad dx}$$

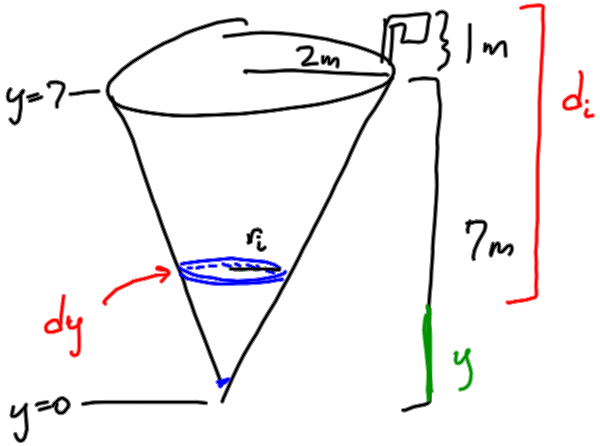
$f(x) =$ Total Weight $-$ rope density $*$ x $= 25 - \frac{1}{6}x$

Weight Rope $= 60 \cdot \frac{1}{6} = 10$ lbs

Weight bucket $= 15$ lbs

Total weight $= 25$ lbs

9. A tank has the shape of a cone, where the radius at the top is 2 m, and the cone is 7 m high. Suppose the water in the tank is 4 m deep, and all the water is pumped out of a spout 1 m above the top of the cone. Write an integral for how much work is done, but do not evaluate. Use ρg for the weight density of water.

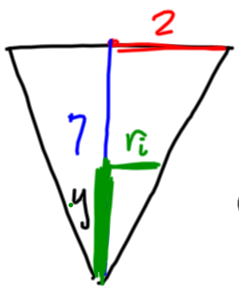


$$A_i = \pi r_i^2 = \pi\left(\frac{2}{7}y\right)^2 = \frac{4\pi y^2}{49}$$

$$d_i = 8 - y$$

$$\boxed{W = \rho g \int_0^4 \frac{4\pi}{49} y^2 (8-y)\, dy}$$

$$W = \rho g \int A_i \cdot d_i \; dy$$



$$\frac{2}{7} = \frac{r_i}{y}$$

$$r_i = \frac{2}{7} y$$

10. Integrate the following

(a) $-\int_0^5 \frac{x}{\sqrt{9-x}} - dx$

u-sub

$u = 9-x \Rightarrow x = 9-u$

$du = -dx$

Bounds: $u(5) = 9-5 = 4$

$u(0) = 9-0 = 9$

$$-\int_9^4 \frac{9-u}{\sqrt{u}}\,du$$

$$\int_4^9 9u^{-1/2} - u^{1/2}\,du$$

(b) $\int \sqrt{x} \ln(x)\, dx$

by parts

$u = \ln x$ $\qquad v = \frac{2}{3} x^{3/2}$

$du = \frac{1}{x}\, dx$ $\qquad dv = \sqrt{x}\, dx$

(c) $\int_0^{\pi/6} \sin^2(2x)\,dx.$

Trig Integral

(d) $\int \sin^3(x)\cos^2(x)\,dx$

Trig Integral

(e) $\int \tan^3(x)\sec^5(x)\,dx$

Trig Integral

(f) $\int_0^1 \frac{x^2}{e^x}\,dx \quad = \quad \int_0^1 x^2 \cdot e^{-x}\,dx$

by parts

Tabular

$u = x^2$	$\xrightarrow{(+)}$	$dv = e^{-x}$
$2x$	$\xrightarrow{(-)}$	$-e^{-x}$
2	$\xrightarrow{(+)}$	e^{-x}
0		$-e^{-x}$

$$-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_0^1$$

$$-e^{-x}\left(x^2+2x+2\right)\Big|_0^1$$

(g) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-16}}$

Trig Sub

$x = 4\sec(\theta)$

$dx = 4\sec\theta\tan\theta\, d\theta$

$$\int \frac{4\sec\theta\tan\theta\, d\theta}{4^2\sec^2\theta\sqrt{16\sec^2\theta - 16}}$$

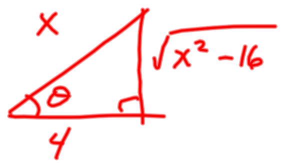
$$\int \frac{4\sec\theta\tan\theta\, d\theta}{4^3\sec^2\theta\tan\theta} = \int \frac{1}{4^2\sec\theta}\,d\theta = \int \frac{\cos\theta}{16}\,d\theta$$

$$\frac{\sin\theta}{16}$$

$$\boxed{\frac{\sqrt{x^2-16}}{16x} + C}$$

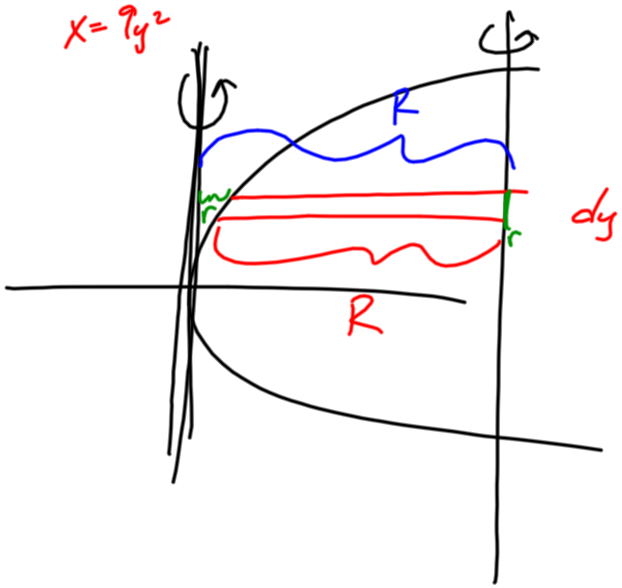
$\frac{x}{4} = \sec\theta$

$\frac{4}{x} = \cos\theta = \frac{\text{adj}}{\text{hyp.}}$



(h) $\int_0^{1/3} \sqrt{4-9x^2}\,dx$

Trig Sub



$$\pi \int_{-1}^{1} [9-9y^2]^2 - 0^2 \, dy$$

(i) $\displaystyle\int \frac{x^3 - 6x^2 + 7x - 9}{x^2 - 3x}\, dx$ Partial Fractions

(j) $\displaystyle\int \frac{8x+12}{x^4+4x^2}\,dx$

Partial Fractions

(k) $\int_1^\infty 5x^2 e^{-x^3}\, dx.$

u-sub

$u = -x^3$

(1) $\displaystyle\int_0^5 \frac{1}{2x-5}\,dx$

$2x-5=0$

$2x=5$

$x=\frac{5}{2}$

$$\cancel{\frac{1}{2}\ln|2x-5|\Big|_0^5}$$

$$\int_0^{5/2}\frac{1}{2x-5}\,dx+\int_{5/2}^{5}\frac{1}{2x-5}\,dx$$

$$\frac{1}{2}\ln|2x-5|\Big|_0^{5/2}$$

$\frac{1}{2}\ln|0|$

$-\infty$

DNE

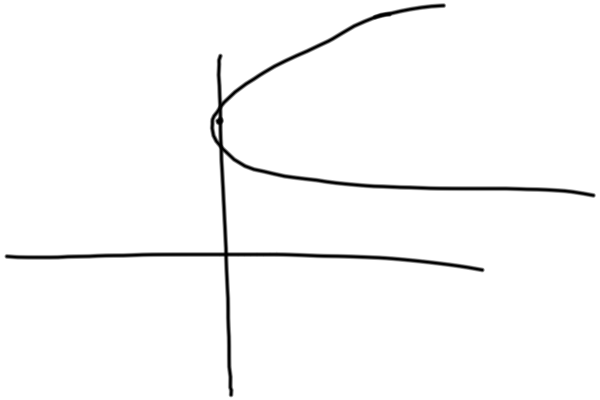
11. Determine if the following integral converges or diverges: $\displaystyle\int_3^\infty \frac{5-\cos(x)}{\sqrt{x-2}}\,dx$

12. Sketch the parametric curves described below.

a.) $x = t^2$, $\underbrace{y = t + 5}$

$t = y - 5$

$x = (y-5)^2$



b.) $x = 3\cos\theta,\ y = 2\sin\theta,\ 0 \le \theta \le \pi$

$\cos\theta = \frac{x}{3}$ $\sin\theta = \frac{y}{2}$

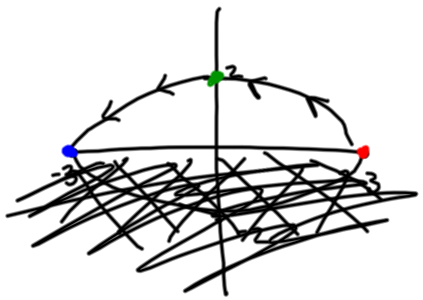
$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

Ellipse! Counterclockwise

θ	x	y
0	3	0
$\pi/2$	0	2
π	-3	0



13. What is the shape of the curve $x = -2 + \sin t$, $y = 5 + \cos t$? In what direction is the curve traced as t increases?

$x + 2 = \sin t$ $\qquad$ $y - 5 = \cos t$

$(x+2)^2 + (y-5)^2 = 1$

Circle centered @ $(-2, 5)$

Clockwise

14. Find the length of the curve $x = 2t^2 + \frac{1}{t}$, $y = 8\sqrt{t}$, $1 \leq t \leq 3$.

$$L = \int \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt}_{ds}$$

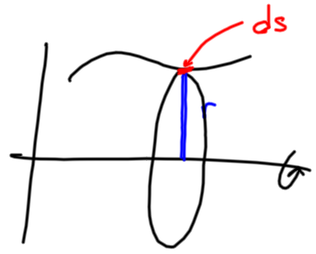
$$\frac{dx}{dt} = 4t - t^{-2} \qquad \frac{dy}{dt} = 4t^{-1/2}$$

$$\ast\left(\frac{dx}{dt}\right)^2 = 16t^2 - 8t^{-1} + t^{-4}\ast \quad + \quad \left(\frac{dy}{dt}\right)^2 = 16t^{-1}$$

$$\int \sqrt{16t^2 + 8t^{-1} + t^{-4}}\;dt$$

$$\boxed{\int_1^3 4t + t^{-2}\;dt}$$

15. Setup an integral, but do not evaluate, for the the surface area obtained by rotating the curve $x = 3t\cos\left(\frac{t}{2}\right)$ and $y = 2e^{t/3}$, $0 \le t \le \frac{\pi}{2}$, about the x-axis, then about the y-axis.



About the x-axis

$r = y = 2e^{t/3}$

$$SA = 2\pi \int r \cdot ds = 2\pi \int r \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

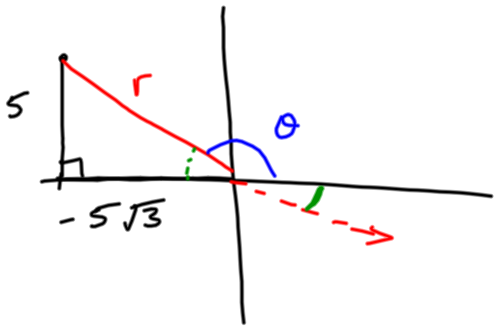
$$\frac{dx}{dt} = \frac{-3t \sin\left(\frac{t}{2}\right)}{2} + 3\cos\left(\frac{t}{2}\right) \qquad \frac{dy}{dt} = \frac{2}{3} e^{t/3}$$

$$\boxed{SA = 2\pi \int_0^{\pi/2} 2e^{t/3} \sqrt{\left[\frac{-3t\sin(t/2)}{2} + 3\cos\left(\frac{t}{2}\right)\right]^2 + \left[\frac{2}{3}e^{t/3}\right]^2}\, dt}$$

About y-axis $\quad r = x = 3t\cos(t/2)$

$$\boxed{SA = 2\pi \int_0^{\pi/2} 3t\cos(t/2) \sqrt{\cdots\cdots}\, dt}$$

16. Write two polar representations, one with $r > 0$ and one with $r < 0$ for the point $(-5\sqrt{3}, 5)$.



$$\sqrt{5^2 + (-5\sqrt{3})^2} = r$$

$$\sqrt{25 + 25(3)}$$

$$\sqrt{100} = r$$

$$r = 10$$

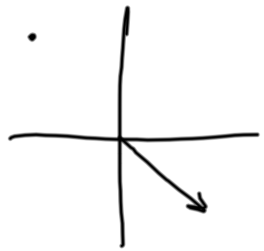
$$\boxed{\left(10, \frac{5\pi}{6}\right)}$$

$$\boxed{\left(-10, \frac{-\pi}{6}\right)}$$

$$\tan\theta = \frac{5}{-5\sqrt{3}}$$

$$\tan\theta = \frac{1/2}{-\sqrt{3}/2} = \frac{\sin\theta}{\cos\theta}$$

$$\theta = \frac{5\pi}{6}$$



17. Write a polar equation for the cartesian equation $x^2 + y^2 = 5x$. What is the shape of the graph?

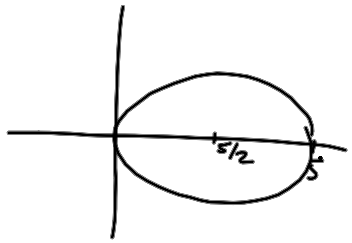
$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$r^2 = 5r\cos\theta$

$\boxed{r = 5\cos\theta}$



$r = a\cos\theta$ or $r = a\sin\theta$

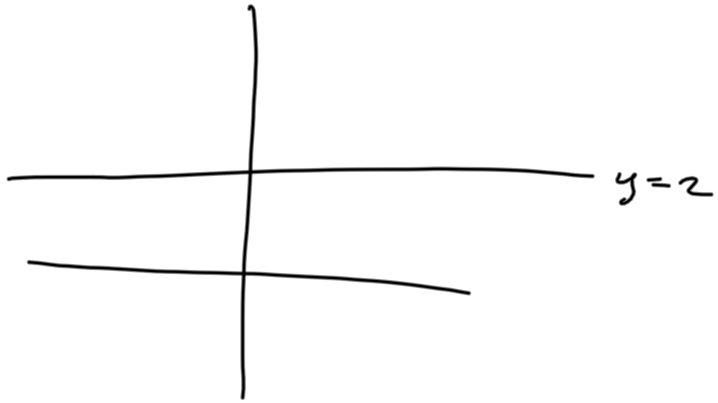
Circles

18. Find a cartesian equation for $r = 2\csc\theta$. What is the shape of the graph?

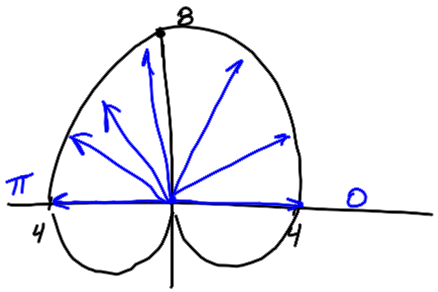
$$r = \frac{2}{\sin\theta}$$

$$r\sin\theta = 2$$

$$y = 2$$



19. Find the area inside the cardioid $r = 4 + 4\sin\theta$ that lies above the x-axis.



$$A = \int_0^{\pi} \frac{1}{2} r^2 \, d\theta$$

$$A = \int_0^{\pi} \frac{1}{2} (4 + 4\sin\theta)^2 \, d\theta$$

$$\frac{1}{2} \int_0^{\pi} 16 + 32\sin\theta + 16\sin^2\theta \, d\theta$$

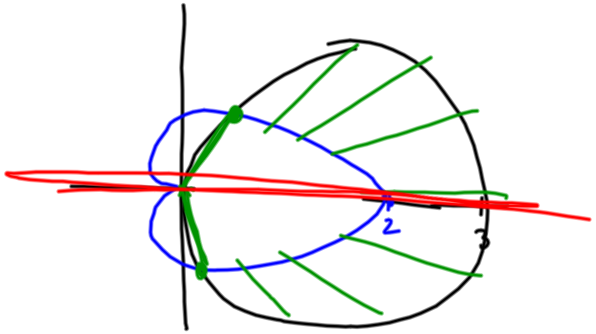
$$\frac{1}{2}\left[16\theta - 32\cos\theta + 16\int_0^{\pi} \frac{1}{2}\left[1 - \cos(2\theta)\right] d\theta\right]$$

$$\frac{1}{2}\left[16\theta - 32\cos\theta + 8\left(\theta - \frac{1}{2}\sin(2\theta)\right)\right]\Big|_0^{\pi}$$

$$\frac{1}{2}\left[16\pi - 32(-1) + 8\left(\pi - \frac{1}{2}(0)\right)\right] - \frac{1}{2}\left[0 - 32(1) + 8\left(0 - \frac{1}{2}(0)\right)\right]$$

$$\boxed{32 + 12\pi}$$

20. Find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.



$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = -\frac{\pi}{3}, \frac{\pi}{3}$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}R^2 - \frac{1}{2}r^2 \, d\theta = \boxed{\frac{1}{2}\int_{-\frac{\pi}{3}}^{\pi/3} [3\cos\theta]^2 - [1+\cos\theta]^2 \, d\theta}$$

or by symmetry

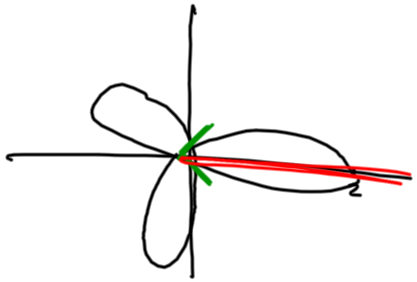
$$\boxed{2 \cdot \frac{1}{2}\int_{0}^{\pi/3} [3\cos\theta]^2 - [1+\cos\theta]^2 \, d\theta}$$

21. Find the area of one petal of the rose $r = 2\cos(3\theta)$.

$r = a \cos(k \cdot \theta)$

If k is odd, there are k petals

If k is even, there are $2k$ petals



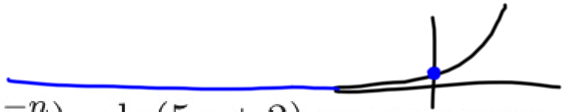
In evaluating / plotting points use increments of $\frac{\pi}{2k}$ $\left(\frac{\pi}{6}\right)$

θ	$r = 2\cos(3\theta)$
0	2
$\frac{\pi}{6}$	0
$\frac{\pi}{3}$	-2
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	2
$\frac{5\pi}{6}$	0
π	-2

$$\boxed{\int_{-\pi/6}^{\pi/6} \frac{1}{2} \left[2\cos(3\theta)\right]^2 \, d\theta}$$

or by symmetry

$$\boxed{2 \int_{0}^{\pi/6} \frac{1}{2} \left[2\cos(3\theta)\right]^2 \, d\theta}$$



22. Determine if the sequence $a_n = \ln(2n + 3e^{-n}) - \ln(5n + 2)$ converges or diverges. If it converges, find its limit.

$$\lim_{n\to\infty} \ln(2n + 3e^{-n}) - \ln(5n+2) = \ln(\infty) - \ln(\infty)$$

$$\lim_{n\to\infty} \ln\left(\frac{2n + 3e^{-n}}{5n+2}\right) = \ln\left(\frac{\infty + \cancel{3(0)}}{\infty + \cancel{2}}\right)$$

$$\boxed{\ln\left(\frac{2}{5}\right)}$$

23. Determine if the sequence $a_n = \dfrac{(-1)^n(5n - n^3)}{2n^3 + 3n^2}$ converges or diverges. If it converges, find its limit.

24. Suppose that $s_n = \dfrac{2n-3}{n+4}$ is the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$.

Find a_4 and $\sum_{n=1}^{\infty} a_n$. $\approx S_\infty = \lim_{n\to\infty} S_n$

$$a_4 = \underbrace{S_4}_{a_1+a_2+a_3+a_4} - \underbrace{S_3}_{a_1+a_2+a_3} = \frac{2(4)-3}{(4)+4} - \frac{2(3)-3}{(3)+4}$$

$$\boxed{\frac{5}{8} - \frac{3}{7}}$$

$$\lim_{n\to\infty} \frac{2n-3}{n+4} = \boxed{2}$$

25. Given the recursive sequence $a_1 = 4$ and $a_{n+1} = 10 - \dfrac{21}{a_n}$ is bounded and increasing, find the limit.

26. Find a general formula for the sequence $\left\{-\frac{7}{4}, \frac{12}{9}, -\frac{17}{16}, \frac{22}{25}, -\frac{27}{36}, \ldots\right\}$. Assume the pattern continues and starts with $n = 1$.

$$\boxed{a_n = \left\{\frac{(-1)^n (5n+2)}{(n+1)^2}\right\}}$$

27. Determine if the series $\sum_{n=1}^{\infty} \left[e^{1/n} - e^{1/(n+2)}\right]$ converges or diverges. If it converges, find its sum.

28. Determine if the series converges absolutely, converges but not absolutely, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{\sqrt{n}}$$

29. Determine if the series converges or diverges and justify your answer.

$$\sum_{n=3}^{\infty} \frac{\cos^2(n) + 4}{n - 2}$$

30. Determine if the series converges or diverges. If it converges, find its sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4^n}{7^{n-1}}$$

31. Determine if the series converges absolutely, converges but not absolutely, or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n^3 + 2n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \left(\sqrt{n} - 3\right)}{n+7}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n^2 + 2n)}{4n^2 - 3n}$

32. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n^2}}{n^3}$ converges or diverges.

33. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 3^{2n}}{(-5)^{n+1}}$ converges or diverges.

34. Determine if the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\arctan n}{3 + 2e^{-n}}$$

$$\lim_{n \to \infty} \frac{\arctan(n)}{3 + 2e^{-n}} = \frac{\arctan(\infty)}{3 + 2e^{-\infty}} = \frac{\pi/2}{3 + 2(0)} = \frac{\pi/2}{3} = \frac{\pi}{6}$$

Series diverges by Test for Divergence

35. Find the radius and interval of convergence for the power series.

(a) $$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2x-5)^n}{3^n \ln n}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 7^{n-1}(x+3)^n}{n^4(n+2)!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!(4x-7)^n}{e^n\sqrt{n}}$

36. Suppose that the series $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence of the following series?

(a) $\sum_{n=0}^{\infty} (-1)^n c_n 3^n$ (b) $\sum_{n=0}^{\infty} c_n 4^n$ (c) $\sum_{n=0}^{\infty} c_n 5^n$ (d) $\sum_{n=0}^{\infty} c_n (-6)^n$ (e) $\sum_{n=0}^{\infty} c_n 9^n$

37. Write the Maclaurin series for the function $\dfrac{5x^4}{5+2x^2}$.

38. Write the Maclaurin series for the function $\ln(6 - 2x^3)$.

39. Write the Maclaurin series for the function $\dfrac{x^2}{(7+x)^2}$.

40. Write the Maclaurin series for $3x^2 \cos\left(\frac{x^3}{3}\right)$.

41. Write the Maclaurin series for $x^3e^{-2x^2}$.

42. Calculate $\int \sin\left(3x^4\right)\, dx$ as a Maclaurin series.

43. Find the sum of the following series: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n} \pi^{2n}}{6^{2n}(2n)!}$

44. Find the Taylor series for the function $f(x) = \dfrac{2}{x^2}$ centered at $a = -5$.

45. Find the second degree Taylor polynomial, $T_2(x)$, for $f(x) = \cos(2x)$ centered at $a = \frac{\pi}{6}$.