Week-in-Review

Exam 1 Review

1. $\displaystyle\int \frac{\cos^3(\ln x)}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \cos^3(\ln x) \cdot \frac{1}{x}\,dx$$

$$\int \cos^3(u)\,du = \int \cos^2(u) \cdot \cos(u)\,du$$

$$= \int [1 - \sin^2(u)] \cos(u)\,du$$

$$= \int [1 - z^2]\,dz$$

$$= z - \frac{1}{3} z^3 + C$$

$$= \sin(u) - \frac{1}{3}\sin^3(u) + C$$

$$= \boxed{\sin(\ln x) - \frac{1}{3}\sin^3(\ln x) + C}$$

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$z = \sin u$

$dz = \cos u\, d\theta$

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2. The force required to hold a spring stretched to a length of 7 m is 5 N. Find the work required to stretch the spring from a length of 4 m to 8 m. The natural length of the spring is 3 m.

1 Find k

Spring Force : $f(x) = kx$

$f(x) = k(7-3)$

$5 = 4k$

$k = \frac{5}{4}$

$f(x) = \frac{5}{4}x$

2 Answer question

$W = \int f(x)\,dx$

$= \int_{4-3}^{8-3} \frac{5}{4} x \, dx$

$\frac{5}{8} x^2 \Big|_1^5$

$\frac{5}{8}(25-1) = \frac{5}{8}(24) = \boxed{15 \text{ J}}$

3. Find the volume of the solid S whose base is bounded by the region $x^2 + 4y^2 = 4$, and cross-sections perpendicular to the y-axis are isosceles trianges with height equal to the base.

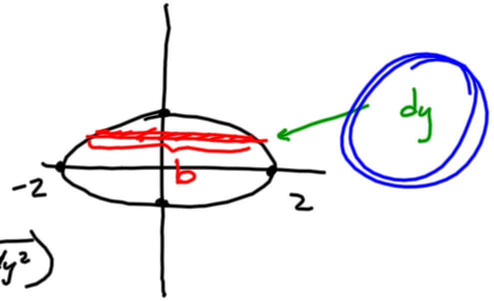
$$x^2 + 4y^2 = 4$$

$$A_i = \frac{1}{2} b h \qquad h = b$$

$$x^2 = 4 - 4y^2$$

$$x = \pm\sqrt{4-4y^2}$$

$$= \frac{1}{2} b \cdot b = \frac{1}{2} b^2$$



$$b = \sqrt{4-4y^2} - \left(-\sqrt{4-4y^2}\right)$$

$$V_i = \frac{1}{2} b^2 \cdot dy$$

$$b = 2\sqrt{4-4y^2}$$

$$V_i = \frac{1}{2}\left(2\sqrt{4-4y^2}\right)^2 dy$$

$$V_i = \frac{1}{2}\left(4\left(4-4y^2\right)\right) dy$$

$$V = \int_{-1}^{1} 8\left(1-y^2\right) dy \quad \text{or} \quad 2\int_0^1 8\left(1-y^2\right) dy$$

$$V_i = 2\left(4-4y^2\right) dy$$

$$V_i = 8\left(1-y^2\right) dy$$

$$16\left[y - \frac{1}{3}y^3\right]_0^1$$

$$(16)\frac{2}{3} = \boxed{\frac{32}{3}}$$

4. Find the area bounded by $x = 3y - y^2$ and $y = -\frac{x}{2}$. $\Rightarrow$ $x = -2y$

$3y - y^2 = 0$

$y(3-y) = 0$

$y = 0 \qquad 3 - y = 0$

$y = 3$

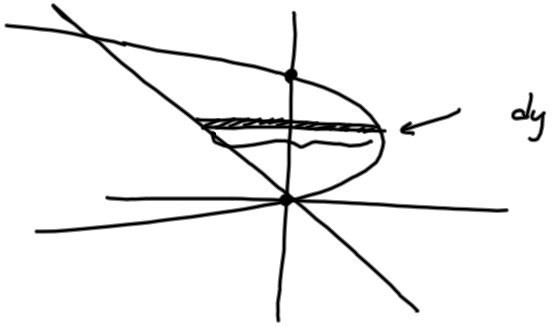
Bounds:

$3y - y^2 = -2y$

$5y - y^2 = 0$

$y(5-y) = 0$

$y = 0 \qquad y = 5$



$h = (3y - y^2) - (-2y) = 5y - y^2$

$$\int_0^5 (5y - y^2)\, dy = \left.\frac{5y^2}{2} - \frac{1}{3}y^3\right|_0^5 = \frac{125}{2} - \frac{125}{3} = \boxed{\frac{125}{6}}$$

5. $\int \frac{\ln x}{\sqrt{x}}\, dx$

$u = \ln x \longrightarrow v = 2x^{1/2}$

$du = \frac{1}{x}\, dx \qquad dv = \frac{1}{\sqrt{x}}\, dx = x^{-1/2}\, dx$

$= \ln(x) \cdot 2x^{1/2} - \int 2x^{1/2} \cdot \frac{1}{x}\, dx$

$- \int 2x^{-1/2}\, dx$

$- 2 \cdot (2) x^{1/2}$

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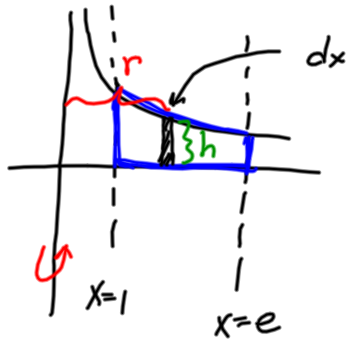
$$\boxed{2\sqrt{x}\ln(x) - 4\sqrt{x} + C}$$

6. The region bounded by $y = \dfrac{1}{x^2}$, $x = 1$, $x = e$, and $y = 0$ is rotated around the y-axis. Find the volume.



$$h = \frac{1}{x^2} - 0 = \frac{1}{x^2}$$

$$r = x - 0 = x$$

$$\int 2\pi r h \, dx$$

$$2\pi \int_1^e x \cdot \frac{1}{x^2} \, dx$$

$$2\pi \int_1^e \frac{1}{x} \, dx = 2\pi \ln|x| \Big|_1^e = 2\pi [\ln(e) - \ln(1)]$$

$$2\pi [1 - 0] = \boxed{2\pi}$$

7. The region bounded by $x + y^2 = 4$ and $x - y = 2$ is rotated around the line $x = -1$. Set up but do not evaluate an integral representing the volume of the solid.

$x = 4 - y^2$ $\qquad$ $x - 2 = y$

$x = y + 2$

Bounds: $4 - y^2 = y + 2$

$0 = y^2 + y - 2$

$0 = (y+2)(y-1)$

$y + 2 = 0 \qquad y - 1 = 0$

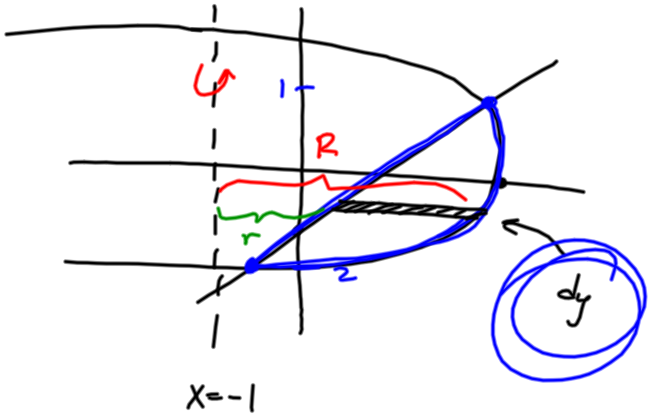
$y = -2 \qquad y = 1$

$R = (4 - y^2) - (-1)$

$= 5 - y^2$

$r = (y+2) - (-1)$

$= y + 3$



$$\boxed{V = \pi \int_{-2}^{1} (5 - y^2)^2 - (y+3)^2 \, dy}$$

8. $\int_0^2 x^2 e^{3x}\,dx$

$u = x^2 \longrightarrow v = \frac{1}{3}e^{3x}$

$du = 2x\,dx \qquad dv = e^{3x}\,dx$

$x^2 \frac{1}{3}e^{3x}\Big|_0^2 + \int_0^2 -2x \cdot \frac{1}{3}e^{3x}\,dx$

$z = -2x \longrightarrow w = \frac{1}{9}e^{3x}$

$dz = -2\,dx \qquad dw = \frac{1}{3}e^{3x}\,dx$

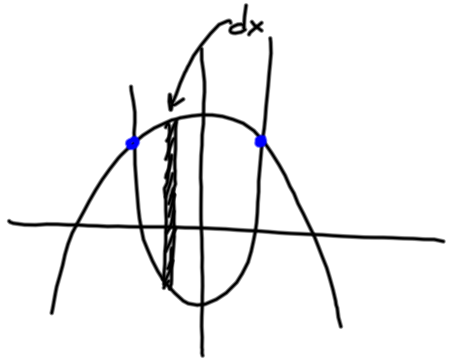
$\frac{x^2 e^{3x}}{3}\Big|_0^2 + -2x\,\frac{1}{9}e^{3x}\Big|_0^2 - \int_0^2 -2\,\frac{1}{9}e^{3x}\,dx$

$\left[\frac{x^2 e^{3x}}{3} - \frac{2x}{9}e^{3x} + \frac{2}{9}\cdot\frac{1}{3}e^{3x}\right]_0^2$

$e^{3x}\left[\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right]\Big|_0^2 = \boxed{e^6\left(\frac{4}{3} - \frac{4}{9} + \frac{2}{27}\right) - e^0\left(0 - 0 + \frac{2}{27}\right)}$

9. Find the area bounded by $y = 7 - x^2$ and $y = 2x^2 - 5$.

$$A = \int_{-2}^{2} \left[(7-x^2) - (2x^2-5) \right] dx$$



or

$$2\int_0^2 \left[(7-x^2) - (2x^2-5) \right] dx$$

$$7-x^2 = 2x^2-5$$

$$0 = 3x^2 - 12$$

$$0 = 3(x^2-4)$$

$$0 = 3(x+2)(x-2)$$

$$x+2=0 \qquad x-2=0$$

$$x=-2 \qquad x=2$$

$$2\int_0^2 -3x^2+12 \; dx$$

$$2\left(-x^3+12x\right)\Big|_0^2$$

$$2(-8+24) - 0$$

$$\boxed{32}$$

10. Set up but do not evaluate an integral for the volume of the solid obtained by rotating the region bounded by $y = x^2 - x$ and $y = 2$ rotated around the line $x = 3$.

$x^2 - x = 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

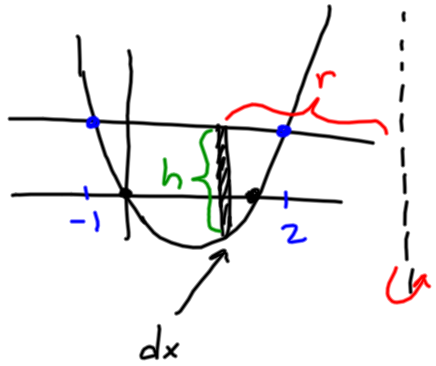
$x = 2 \quad x = -1$

$0 = x^2 - x$

$0 = x(x-1)$

$x = 0 \quad x - 1 = 0$

$x = 1$



$h = 2 - (x^2 - x) = 2 + x - x^2$

$r = 3 - x$

$$\boxed{V = 2\pi \int_{-1}^{2} (3-x)(2+x-x^2)\, dx}$$

11. $\displaystyle\int \frac{x^3}{(x^2+1)^8}\,dx$

$u = x^2+1 \Rightarrow x^2 = u-1$

$du = 2x\,dx$

$$\frac{1}{2}\int \frac{x^2 \cdot 2x}{(x^2+1)^8}\,dx$$

$$\frac{1}{2}\int \frac{u-1}{u^8}\,du = \frac{1}{2}\int \frac{u}{u^8} - \frac{1}{u^8}\,du$$

$$\frac{1}{2}\int u^{-7} - u^{-8}\,du$$

$$\frac{1}{2}\left(\frac{u^{-6}}{-6} - \frac{u^{-7}}{-7}\right) + C$$

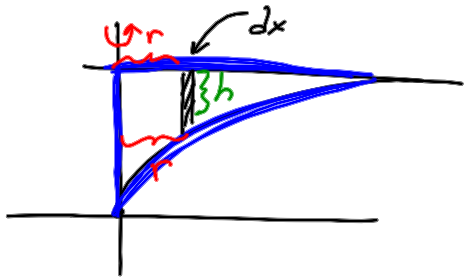
$$\frac{1}{2}\left(\frac{(x^2+1)^{-6}}{-6} + \frac{(x^2+1)^{-7}}{7}\right) + C$$

$$\boxed{\frac{1}{14(x^2+1)^7} - \frac{1}{12(x^2+1)^6} + C}$$

12. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$ around the y-axis.

$h = 2 - \sqrt{x}$

$r = x - 0 = x$



Bounds

$2 = \sqrt{x}$ $\quad$ $0 = \sqrt{x}$

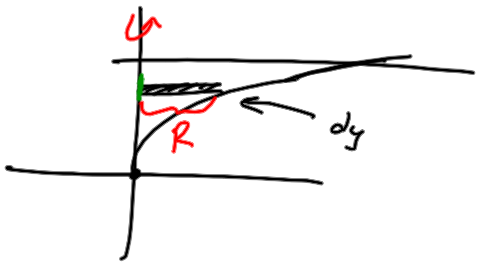
$4 = x$ $\quad$ $0 = x$

$$\boxed{V = 2\pi \int_0^4 x\,(2 - \sqrt{x})\,dx}$$

$y = \sqrt{x}$

$x = y^2$

$R = y^2 - 0 = y^2$



$$\boxed{V = \pi \int_0^2 (y^2)^2\,dy}$$

13. $\int \tan^6 x \sec^4(x)\, dx$

$\sin^2(x) + \cos^2(x) = 1$

$\sin^2(x) = 1 - \cos^2(x)$

$\int \tan^6(x)\, \sec^2(x)\, \sec^2(x)\, dx$

$\tan^2(x) + 1 = \sec^2(x)$

$\tan^2(x) = \sec^2(x) - 1$

$\int \tan^6(x)\, [\tan^2(x) + 1]\, \sec^2(x)\, dx$

$u = \tan(x)$

$du = \sec^2(x)\, dx$

or

$\cancel{u = \sec(x)}$

$\cancel{du = \sec(x)\tan(x)\, dx}$

$\int u^6 (u^2 + 1)\, du$

$\int u^8 + u^6\, du$

$\frac{1}{9}u^9 + \frac{1}{7}u^7 + C = \boxed{\frac{1}{9}\tan^9(x) + \frac{1}{7}\tan^7(x) + C}$

14. $\int_0^{\pi/6} \sin^2(5x)\,dx$

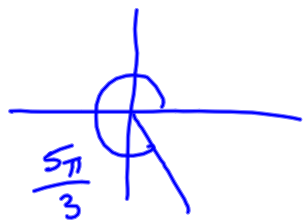
Double Angle Formula

$$\sin^2(\theta) = \frac{1}{2}\left[1 - \cos(2\theta)\right]$$

$$\cos^2(\theta) = \frac{1}{2}\left[1 + \cos(2\theta)\right]$$

$$\int_0^{\pi/6} \frac{1}{2}\left[1 - \cos(10x)\right]\,dx$$

$$\frac{1}{2}\left[x - \frac{1}{10}\sin(10x)\right]\Big|_0^{\pi/6}$$



$$\frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{10}\sin\left(\frac{10\pi}{6}\right)\right) - \frac{1}{2}\left(0 - \frac{1}{10}\sin(0)\right)$$

$$\frac{\pi}{12} - \frac{1}{20}\left(\frac{-\sqrt{3}}{2}\right) - 0 = \boxed{\frac{\pi}{12} + \frac{\sqrt{3}}{40}}$$

15. Find $\int e^x \sin(8x)\,dx$

$u = \sin(8x) \qquad v = e^x$

$du = 8\cos(8x)\,dx \qquad dv = e^x\,dx$

$\sin(8x)\,e^x - \int 8e^x \cos(8x)\,dx$

$+ \int -8\cos(8x)\, e^x\,dx$

$z = -8\cos(8x) \qquad w = e^x$

$dz = 64\sin(8x) \qquad dw = e^x\,dx$

$\sin(8x)\,e^x + -8\cos(8x)\,e^x - \int 64\sin(8x)\,e^x\,dx$

$\int e^x \sin(8x)\,dx = e^x\sin(8x) - 8e^x\cos(8x) - 64\int e^x\sin(8x)\,dx$

$+64\int e^x \sin(8x)\,dx \qquad\qquad + 64\int e^x\sin(8x)\,dx$

$65\int e^x\sin(8x)\,dx = e^x\sin(8x) - 8e^x\cos(8x)$

$$\boxed{\int e^x \sin(8x)\,dx = \frac{e^x\sin(8x) - 8e^x\cos(8x)}{65} + C}$$

16. A bucket attached to a 20 pound rope is used to draw water out of an 80 ft well. The bucket weighs 1 pound and holds 26 pounds of water. How much work is done in drawing up one full bucket of water?

$W = \int f(x)\,dx$

$f(x) =$ The force (lbs) remaining on the rope after x feet have been pulled up.

$f(x) =$ Total weight $-$ (rope density) $\cdot\, x$

Rope density $= \frac{\text{lbs}}{\text{ft.}} = \frac{20}{80} = \frac{1}{4}\,\frac{\text{lbs}}{\text{ft.}}$

$$= \big(\underbrace{20}_{\text{rope}} + \underbrace{1}_{\text{bucket}} + \underbrace{26}_{\text{water}}\big) - \frac{1}{4}x$$

$f(x) = 47 - \frac{1}{4}x$

$$W = \int_0^{80} 47 - \frac{1}{4}x \; dx = \boxed{47x - \frac{1}{8}x^2 \Big|_0^{80}} \text{ ft-lbs}$$

17. Consider the region R bounded by $y = x^3$, $y = 8$, and $x = 0$. Suppose a tank is in the shape of the region R revolved around the y-axis, and the units are measured in meters. If the tank is filled with water to a depth of 3 m, set up but do not evaluate an integral that gives the work done in pumping all the water out of a 1 m high spout. Use ρg for the weight density of water.

$W_i = F_i \cdot d_i$

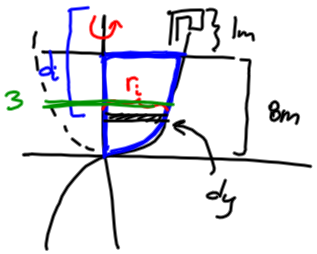
$F_i = M_i \cdot a$

$M_i = \rho \cdot V_i$

$V_i = A_i \cdot dy$

$A_i =$ changes w/ question

$W_i = \rho g \, \pi y^{2/3} (9-y) \, dy$



$y = x^3$

$x = y^{1/3}$

$F_i = \rho g \pi y^{2/3} \, dy$

$M_i = \rho \pi y^{2/3} \, dy$

$V_i = \pi y^{2/3} \, dy$

$A_i = \pi r_i^2 = \pi y^{2/3}$

$r_i = y^{1/3}$

$d_i = 9 - y$

$W = \int W_i = \int_a^b \rho g \, A_i \, d_i \, dy$

$$\boxed{W = \rho g \int_0^3 \pi y^{2/3} (9-y) \, dy}$$